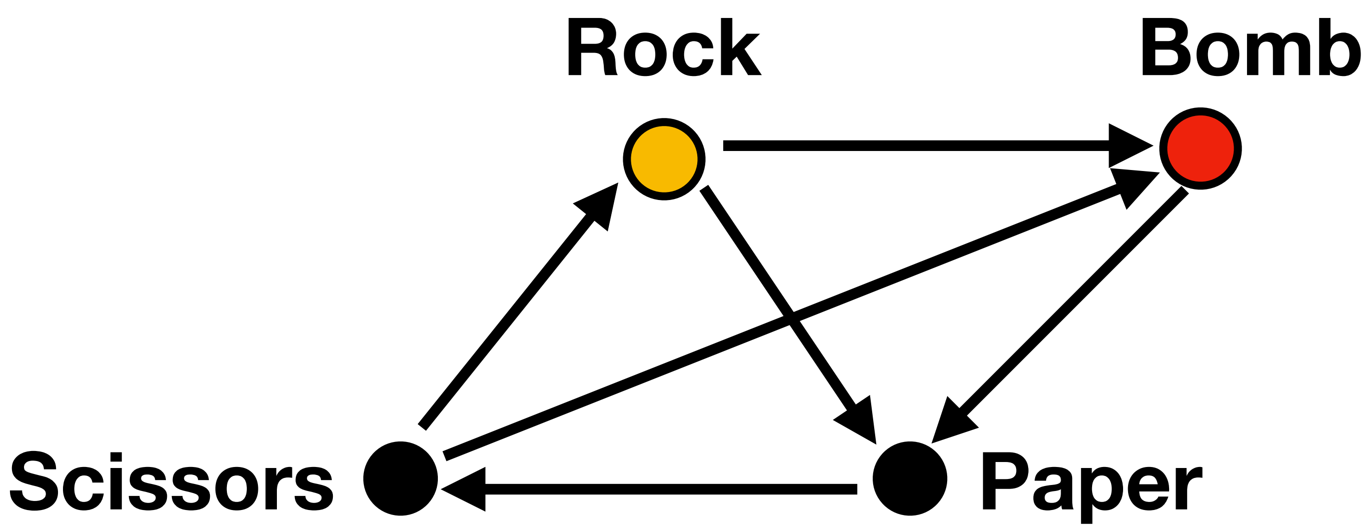
Graphical domination and inhibitory control in recurrent networks



Carina Curto, Brown University

NITMB MathBio Convergence Conference

August 13, 2025

Motivating ideas

1. The brain is a dynamical system. (“The brain is a computer.”)

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2. How does connectivity shape dynamics?

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4. Network motifs can be composed as dynamic building blocks with predictable properties.

Motivating ideas

1. The brain is a dynamical system. ("The brain is a computer.")
2. How does connectivity shape dynamics?
3. By studying ANNs that are dynamical systems, we can generate hypotheses about the dynamic meaning/role of various network motifs.
4. Network motifs can be composed as dynamic building blocks with predictable properties.
5. One network (by architecture/connectivity) is really many networks in the presence of neuromodulation or external control.

Plan of the talk

- Brief intro to TLNs, CTLNs, and gCTLNs
- Fixed points and attractors and graph rules
- Domination
- Dominoes and inhibitory control
- E-I TLNs
- Domination-reduction in connectomes

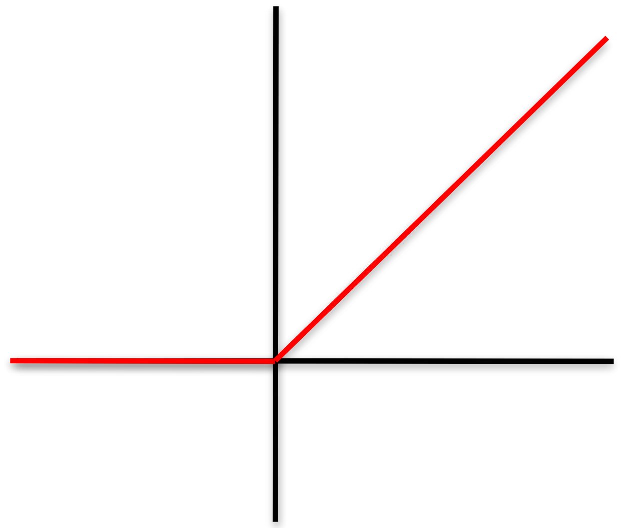
TLNs — nonlinear recurrent network models

Threshold-linear network dynamics:

$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + b_i\right]_+$$

W is an $n \times n$ matrix

$b \in \mathbb{R}^n$



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TLNs — nonlinear recurrent network models

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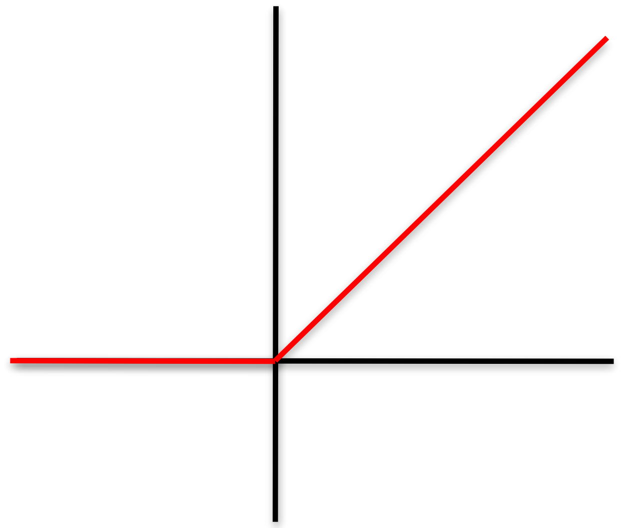

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Basic Question: Given (W,b), what are the network dynamics?

TLNs — nonlinear recurrent network models

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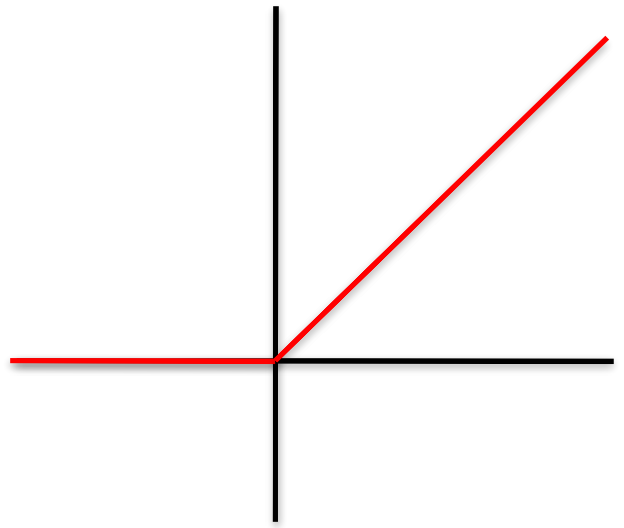


The TLN is defined by (W, b)

Linear network dynamics:

$$\frac{dx}{dt} = Ax + b$$

A is an $n \times n$ matrix

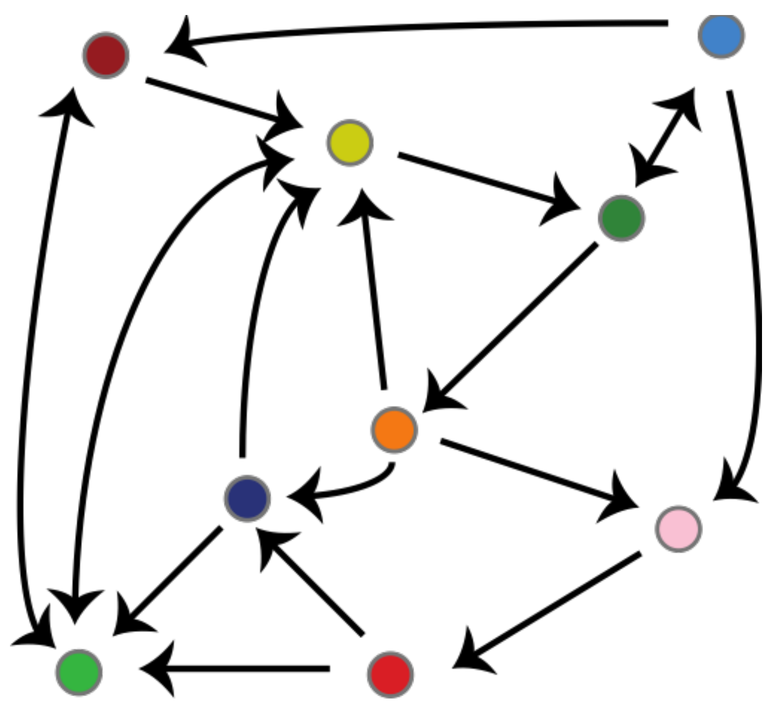
$b \in \mathbb{R}^n$

Long-term behavior is easy to infer from eigenvalues, eigenvectors — linear algebra tells us everything.

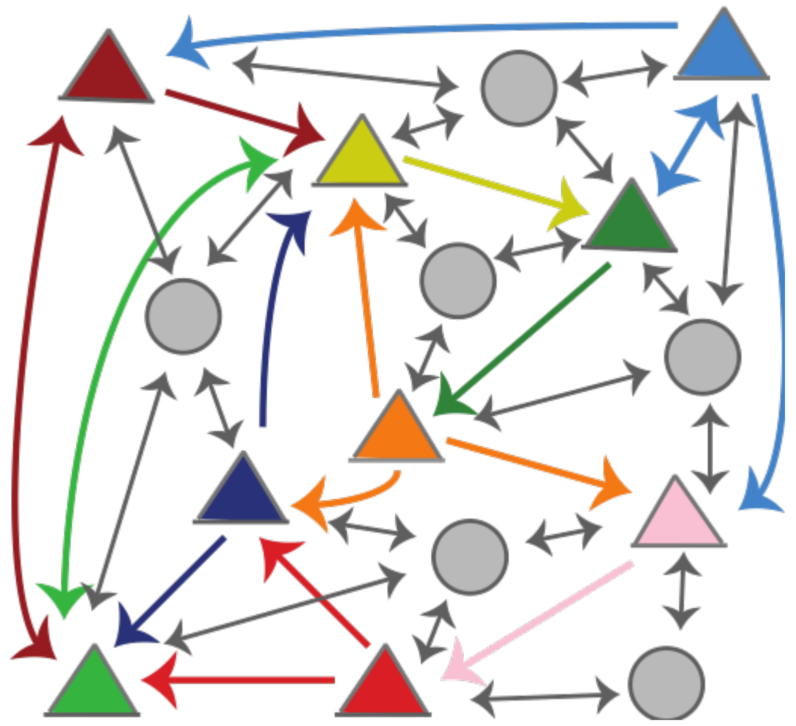
Basic Question: Given (W,b), what are the network dynamics?

The most special case: Combinatorial Threshold-Linear Networks (CTLNs)

graph G



Idea: network of excitatory and inhibitory cells



TLN dynamics:

$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + \theta\right]_+$$

Graph G determines the matrix W

$$W_{ij} = \begin{cases} 0 & \text{if } i = j \\ -1+\varepsilon & \text{if } i \leftarrow j \text{ in } G \\ -1-\delta & \text{if } i \not\leftarrow j \text{ in } G \end{cases}$$

parameter constraints:

$$\delta > 0 \qquad \theta > 0 \qquad 0 < \varepsilon < \frac{\delta}{\delta+1}$$

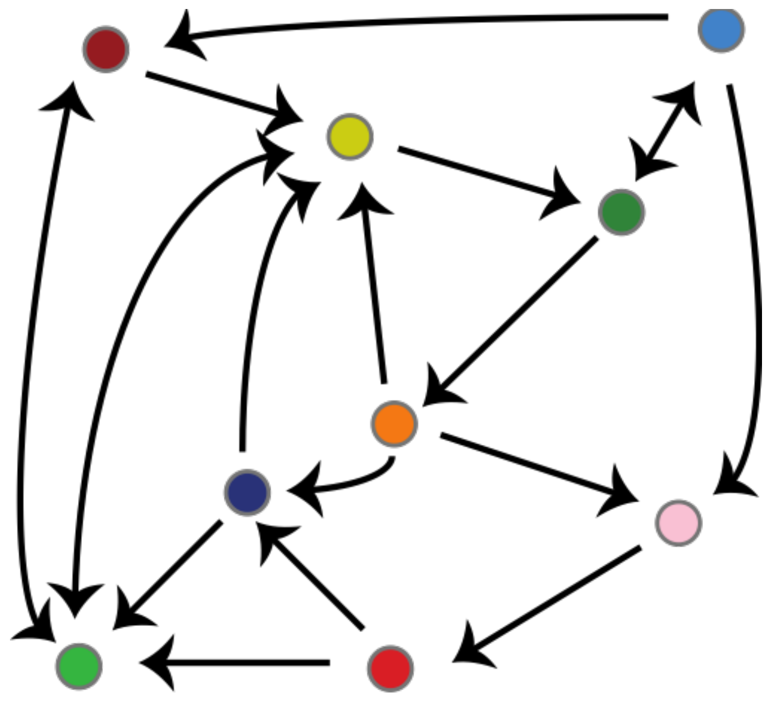
The graph encodes the pattern of weak and strong inhibition

Think: generalized WTA networks

For fixed parameters,
only the graph changes –
isolates the role of connectivity

Less special: generalized Combinatorial Threshold-Linear Networks (gCTLNs)

graph G



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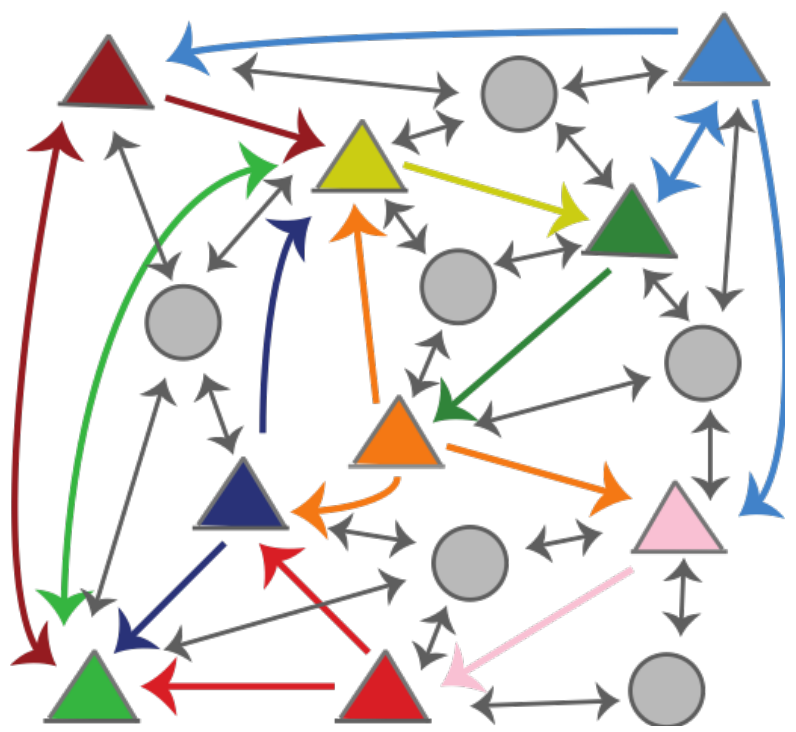


TLN dynamics:

$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + \theta\right]_+$$

The gCTLN is defined by a graph G and two vectors of parameters: ε, δ

$$W_{ij} = \begin{cases} -1+\varepsilon_j & \text{if } j \to i, \\ -1-\delta_j & \text{if } j \not\to i, \\ 0 & \text{if } i = j. \end{cases}$$

weak inhibition

strong inhibition

parameter constraints:

$$\varepsilon_j, \delta_j > 0, \quad \varepsilon_j < \frac{\delta_j}{\delta_j + 1}$$

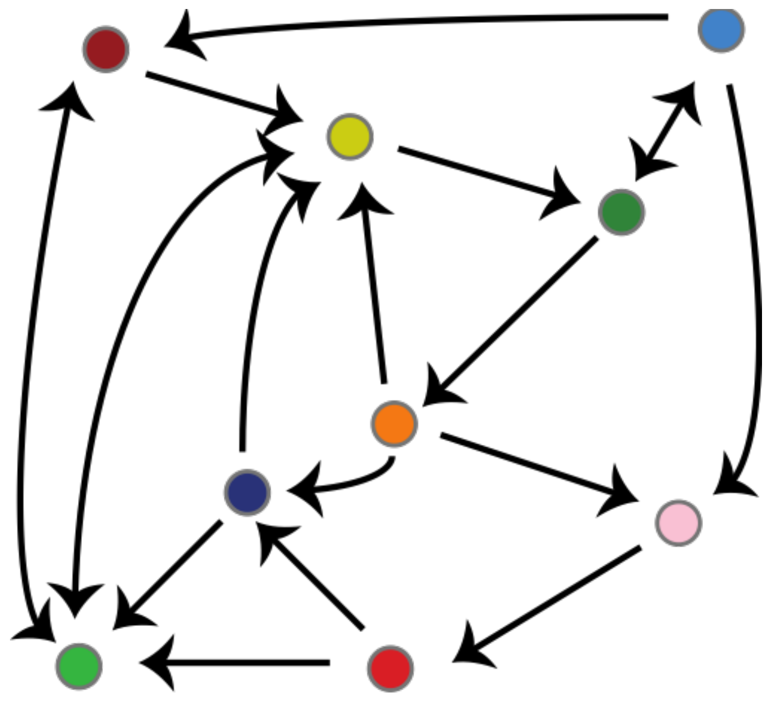
The graph encodes the pattern of weak and strong inhibition

$b_i = \theta > 0$ for all neurons

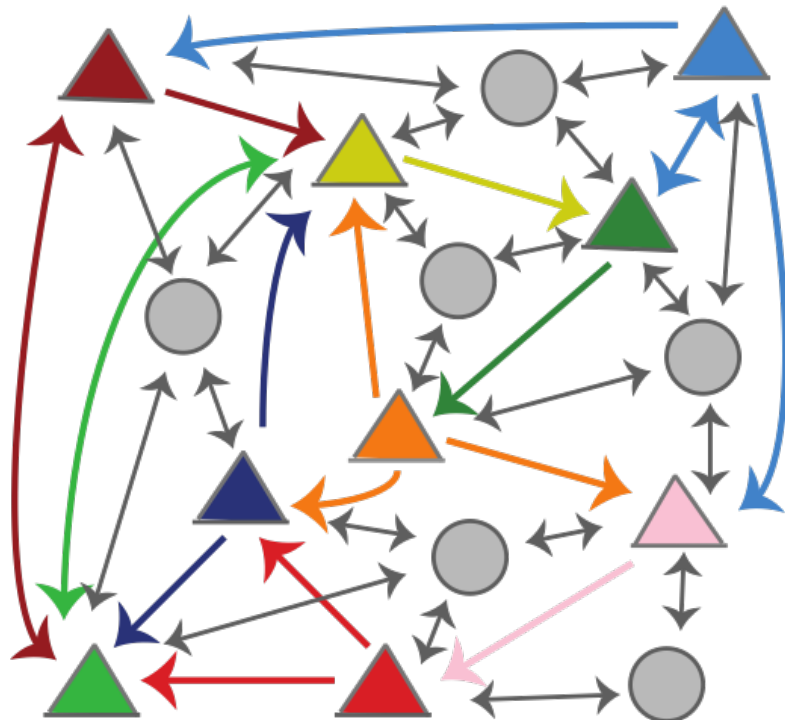
(default is uniform across neurons, constant in time)

Less special: generalized Combinatorial Threshold-Linear Networks (gCTLNs)

graph G



Idea: network of excitatory and inhibitory cells



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weak inhibition

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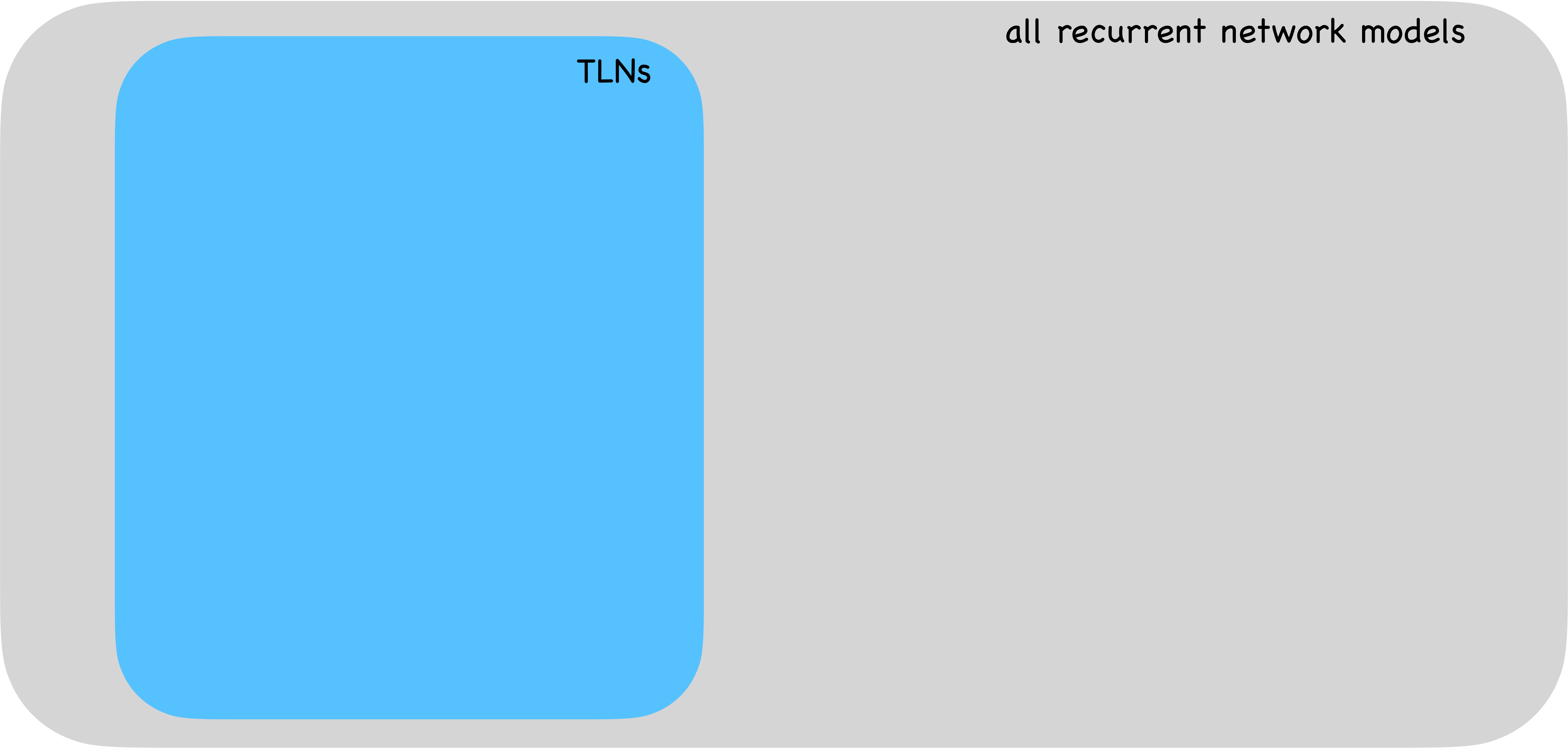
CTLNs

gCTLNs

Special case: if the parameters ε_j, δ_j are the same for all neurons, we have a CTLN.

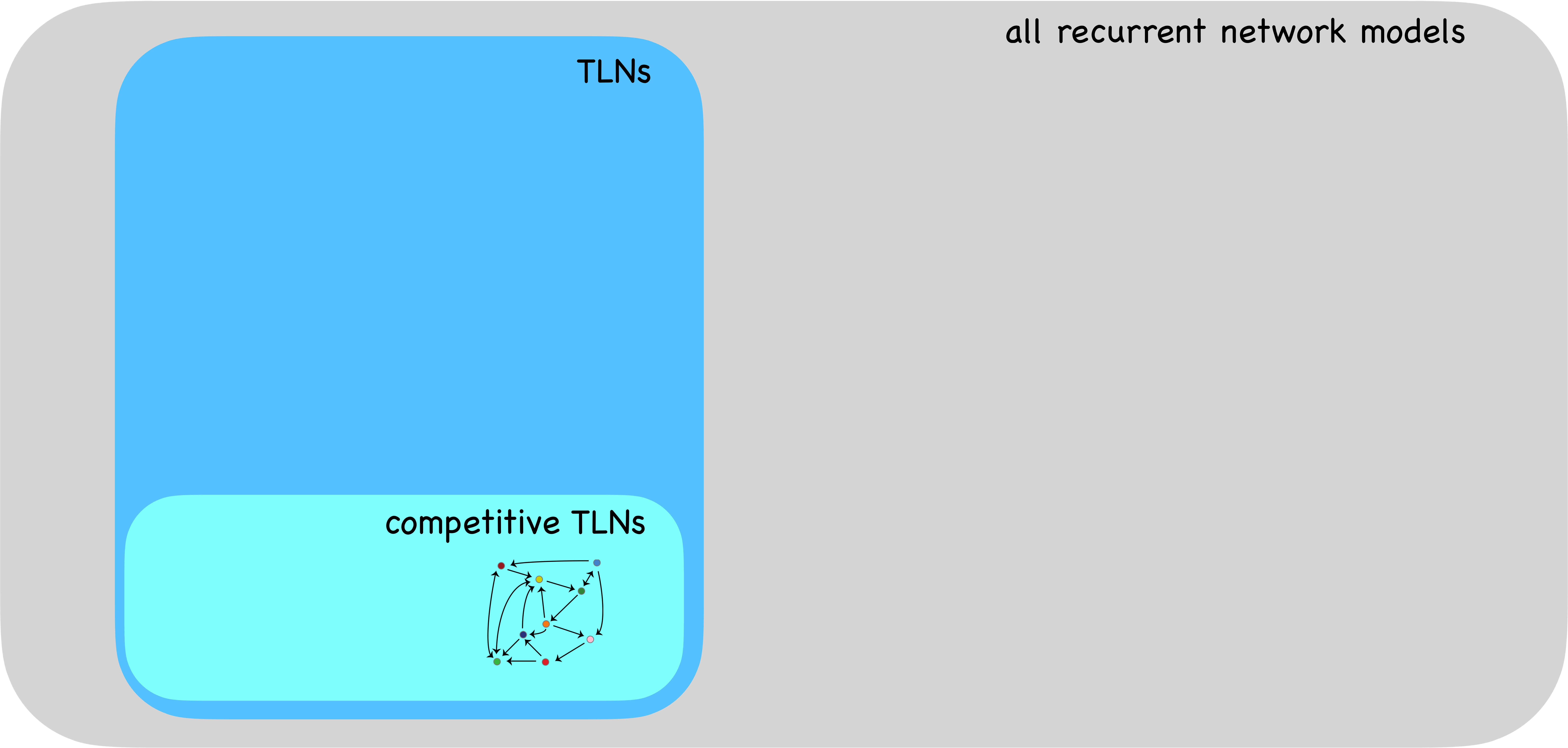
TLNs, CTLNs, and gCTLNs



all recurrent network models

TLNs

TLNs, CTLNs, and gCTLNs

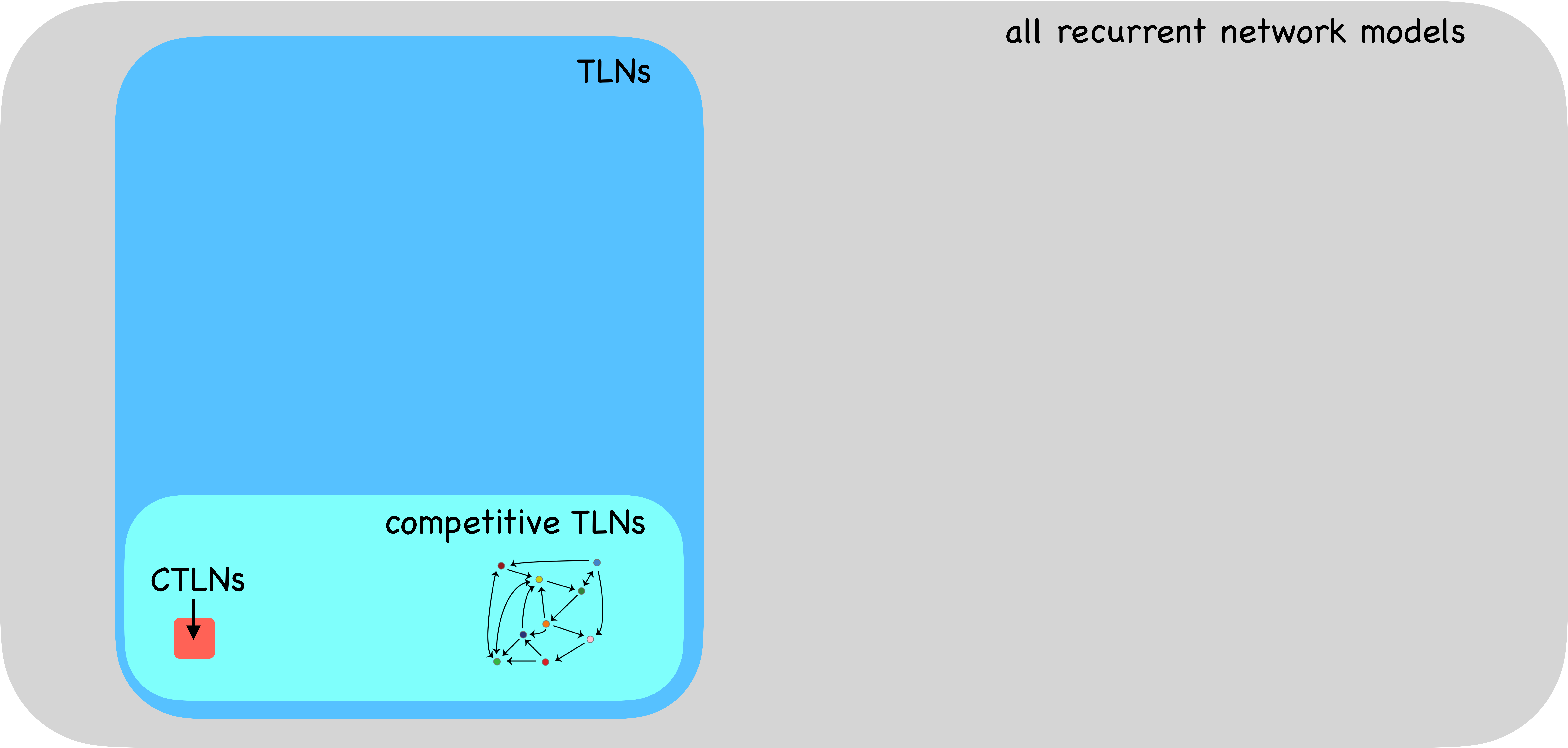


all recurrent network models

TLNs

competitive TLNs

TLNs, CTLNs, and gCTLNs



all recurrent network models

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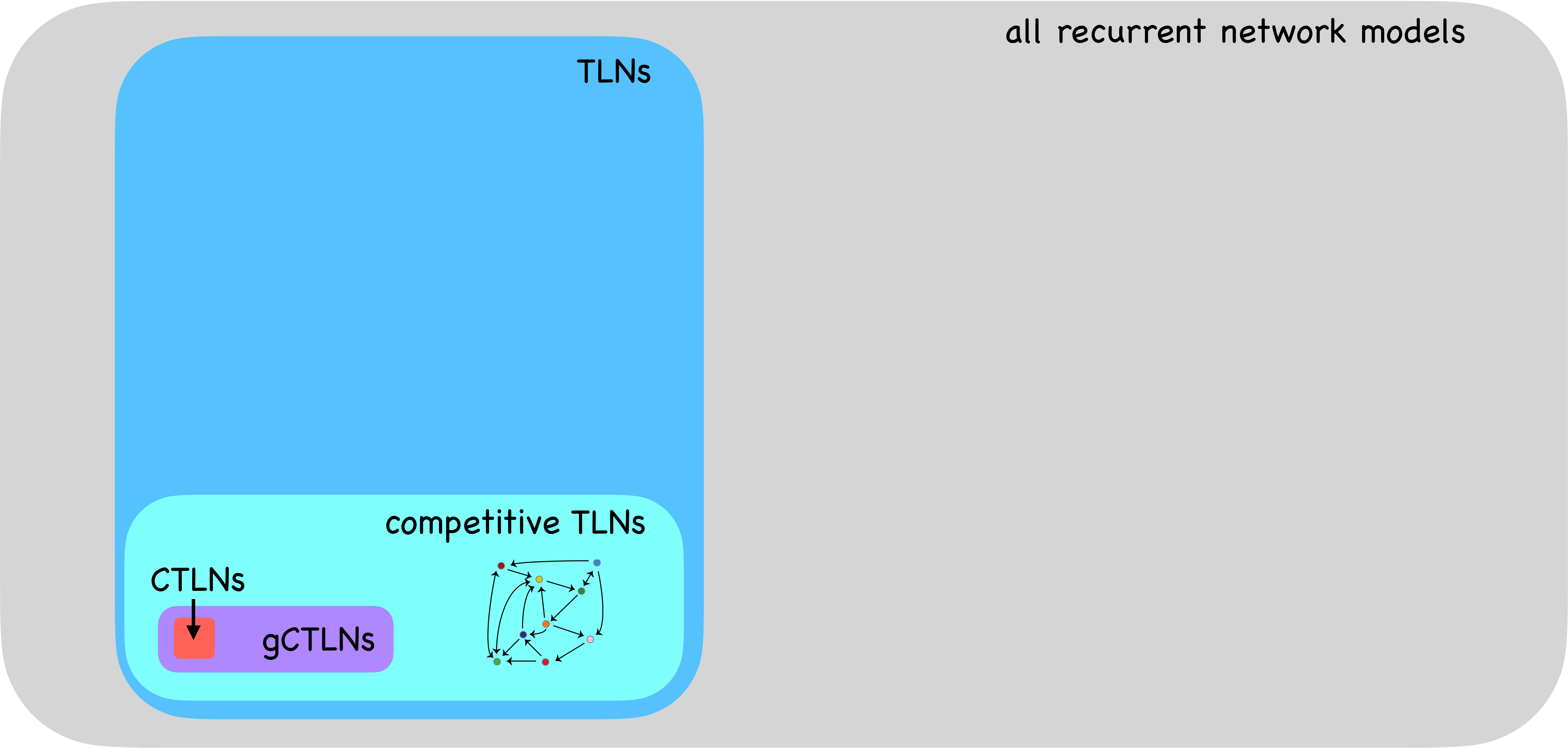

all recurrent network models

TLNs

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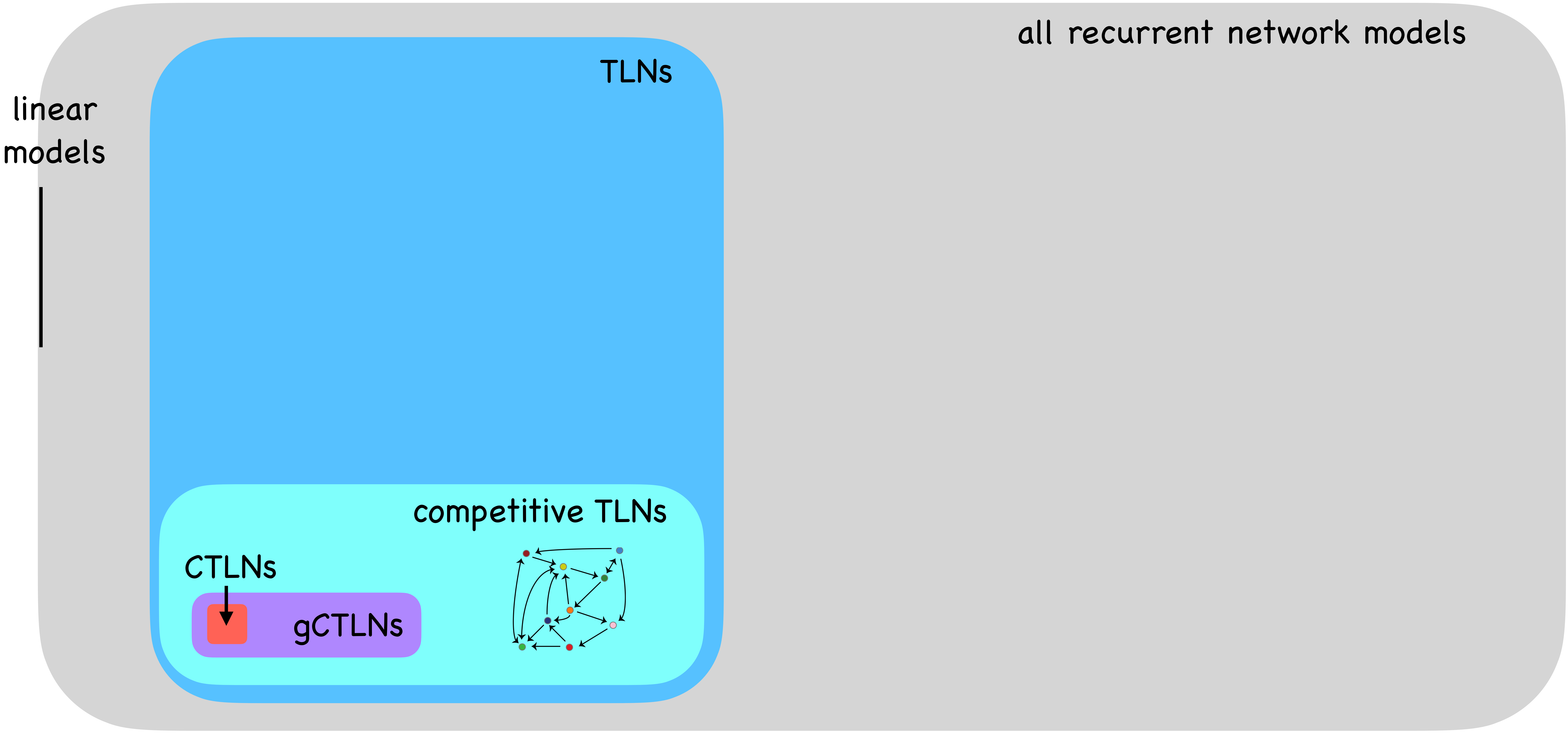
TLNs, CTLNs, and gCTLNs



all recurrent network models

TLNs

linear models

competitive TLNs

CTLNs

gCTLNs

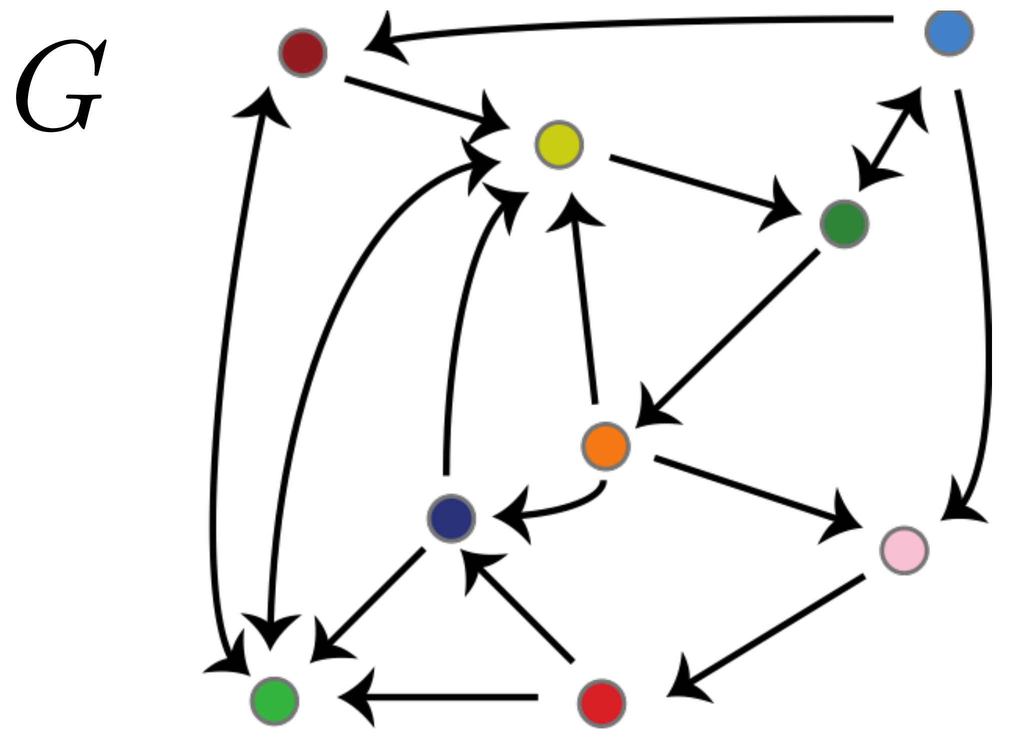
TLNs, CTLNs, and gCTLNs

1. Display rich nonlinear dynamics: multistability, limit cycles, chaos…

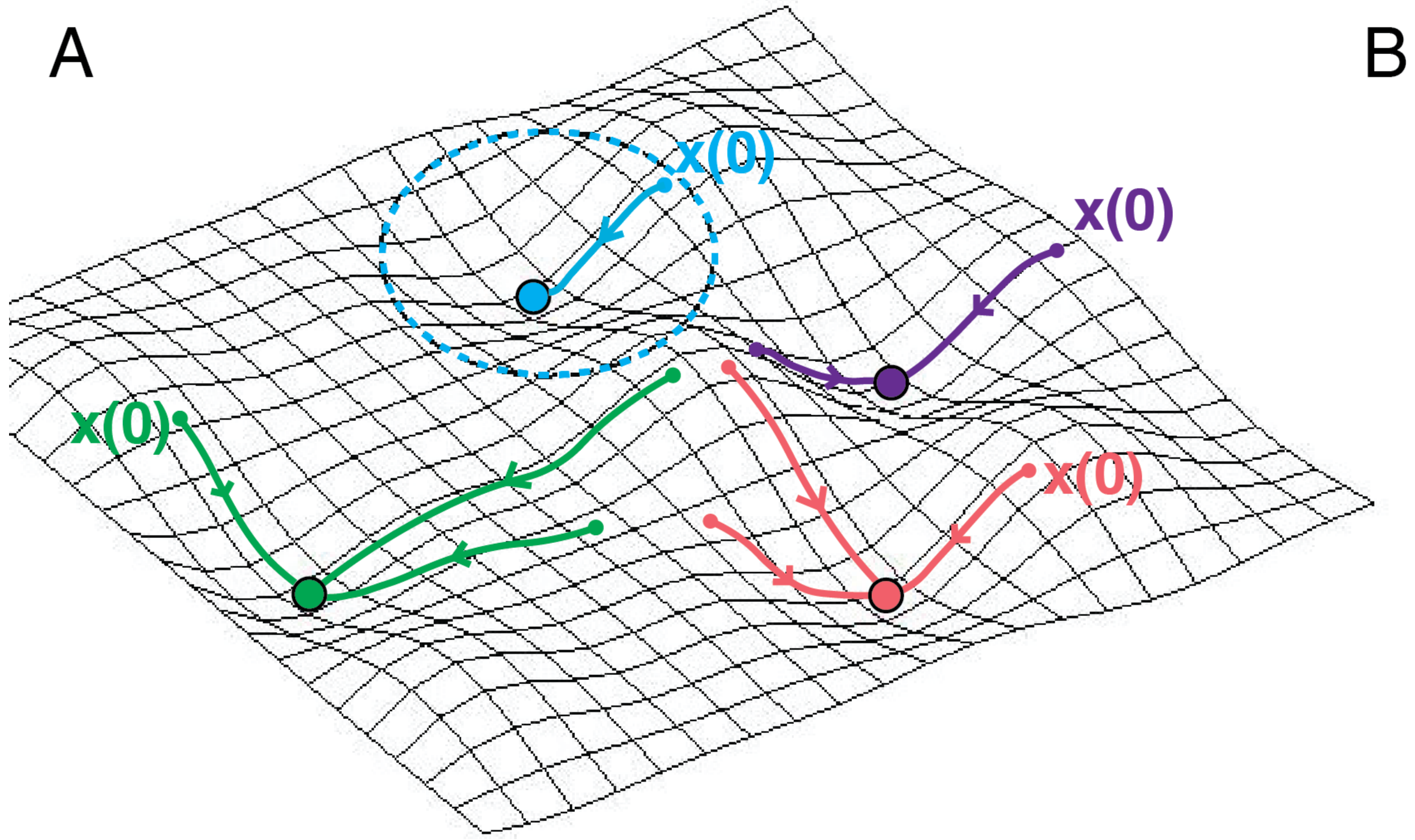
2. Mathematically tractable: we can prove theorems directly connecting graph structure to dynamics.

3. Both stable and unstable fixed points play a critical role in shaping the dynamics (the vector field).

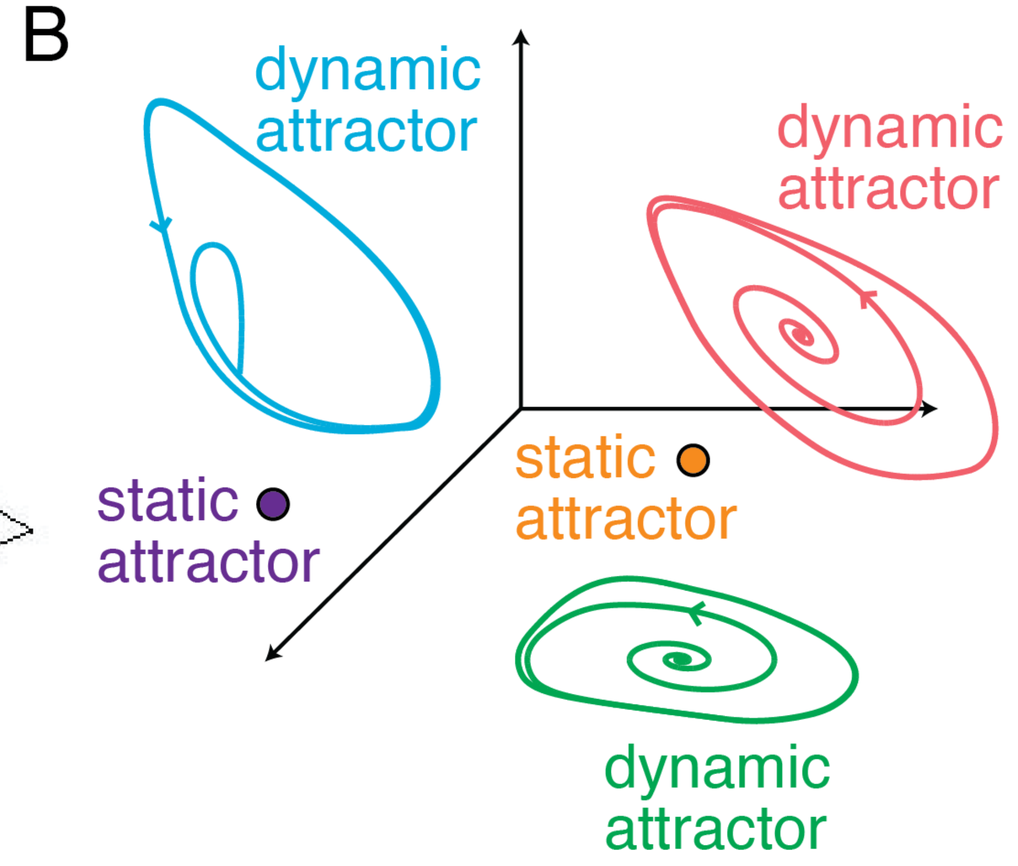
graph structure



static attractors (fixed pts)



dynamic attractors
(correspond to certain unstable fixed pts)

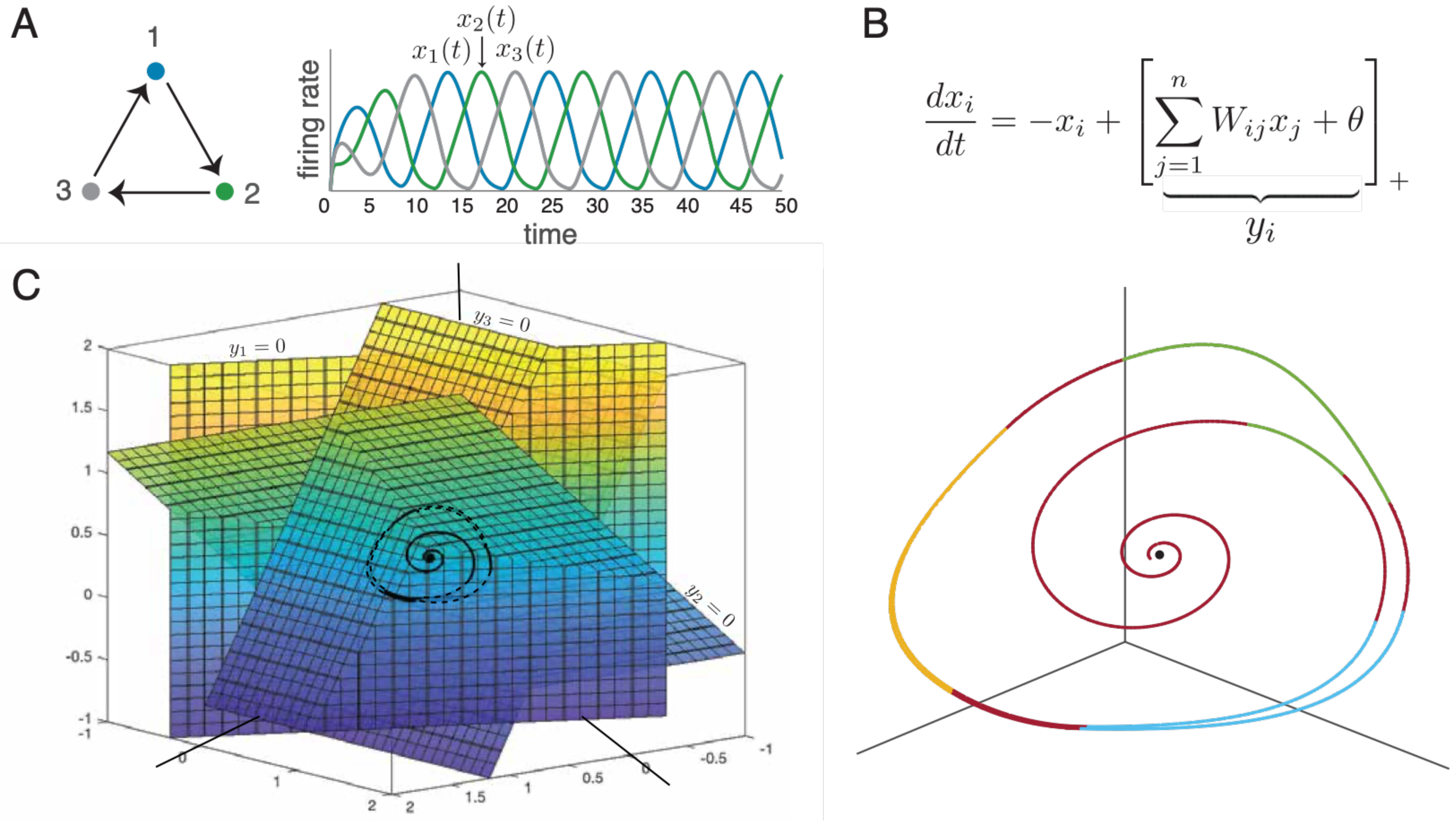


$$\mathrm{FP}(G) = \mathrm{FP}(G, \varepsilon, \delta) = \{ \text{ fixed points (stable and unstable) } \}$$

Curto & Morrison, Notices of the AMS 2023 (review paper)

Theorem: oriented graphs with no sinks

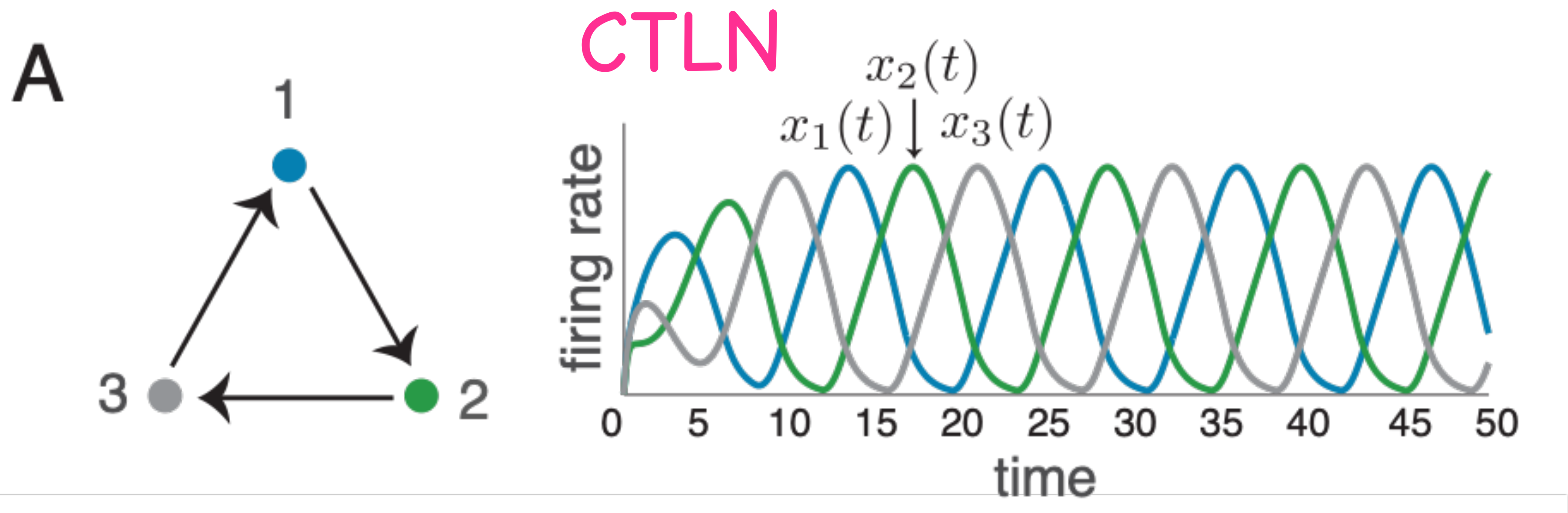
Theorem. If G is an oriented graph with no sinks, then the network has no stable fixed points (but bounded activity).



$$\frac{dx_i}{dt} = -x_i + \underbrace{\left[\sum_{j=1}^{n} W_{ij}x_j + \theta\right]_+}_{y_i}$$

Existence of such limit cycles was established in Bel, Cobiaga, Reartes, and Rotstein, SIADS 2022.

How does the 3-cycle oscillation change for gCTLNs?



B

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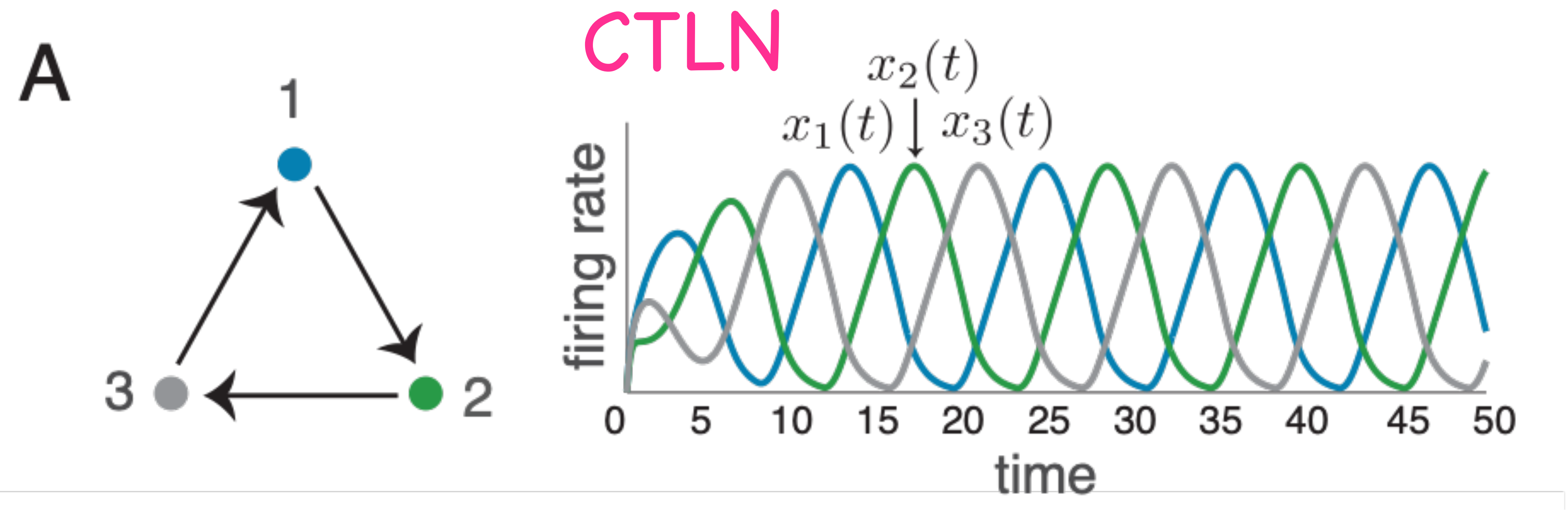

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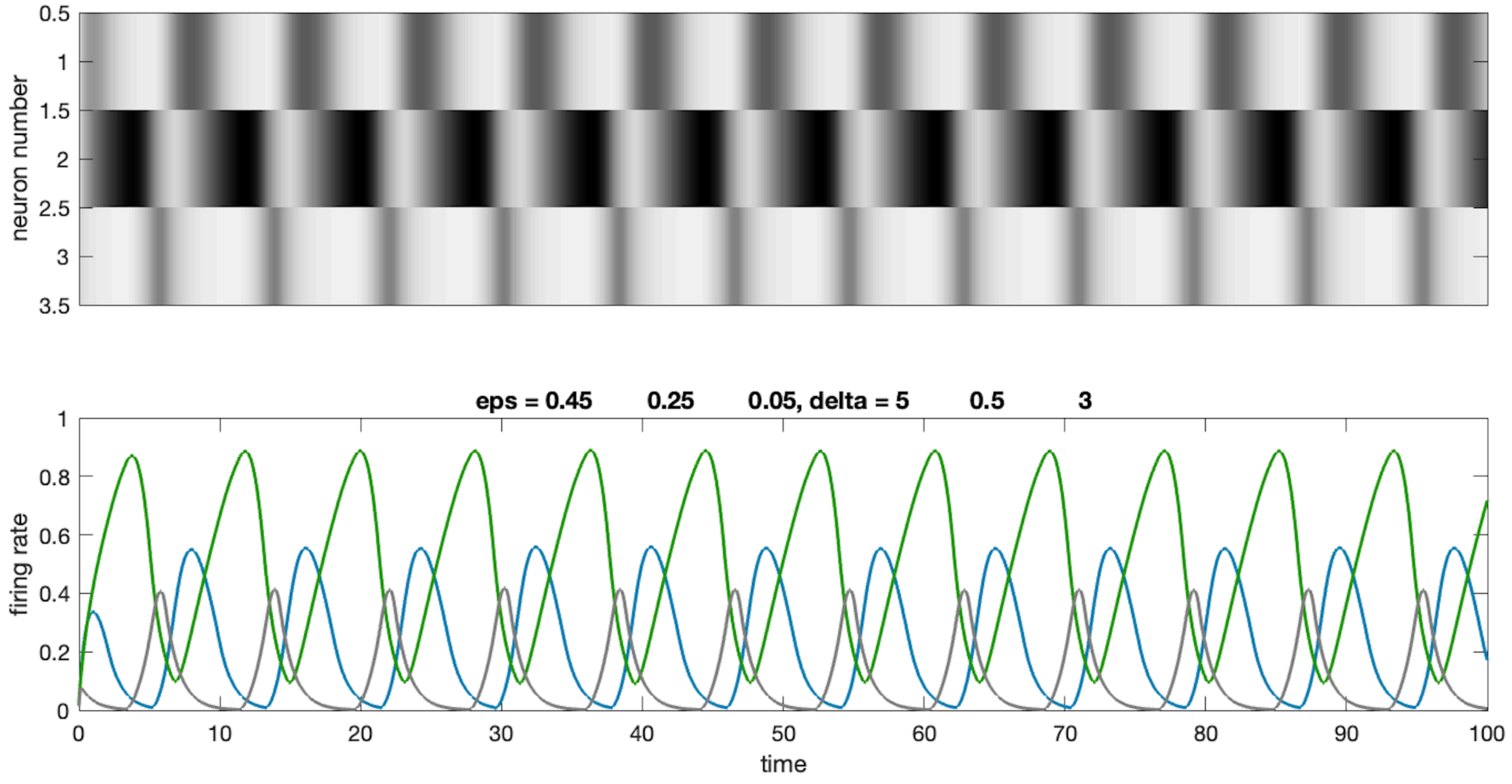
gCTLN #1

unstable fixed point: [0.0812, 0.5481, 0.1018]



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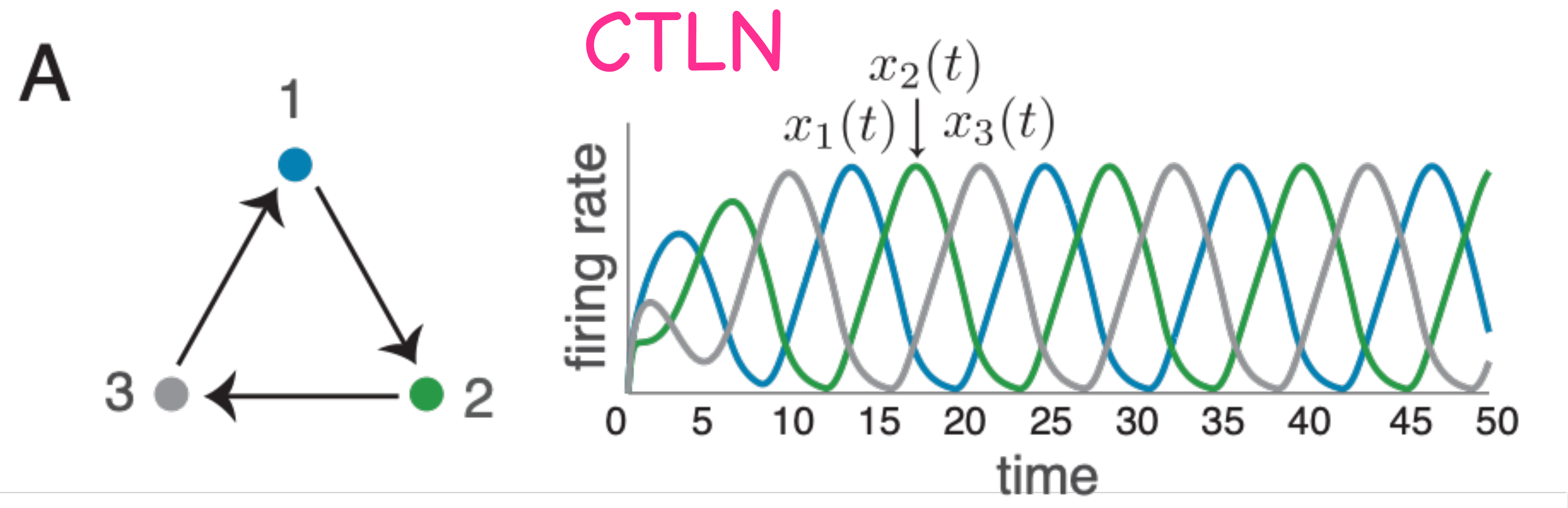

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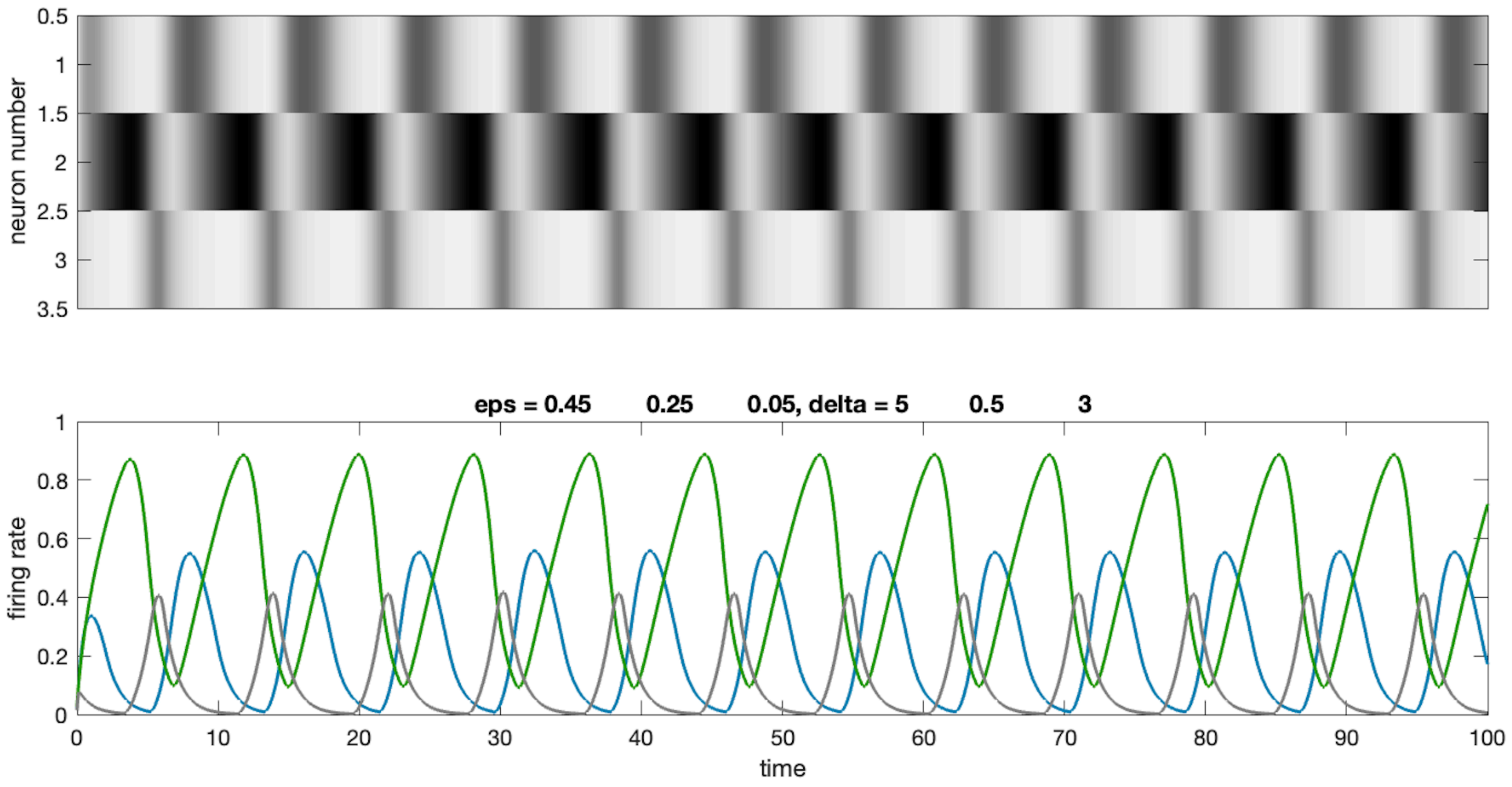
gCTLN #1

unstable fixed point: [0.0812, 0.5481, 0.1018]

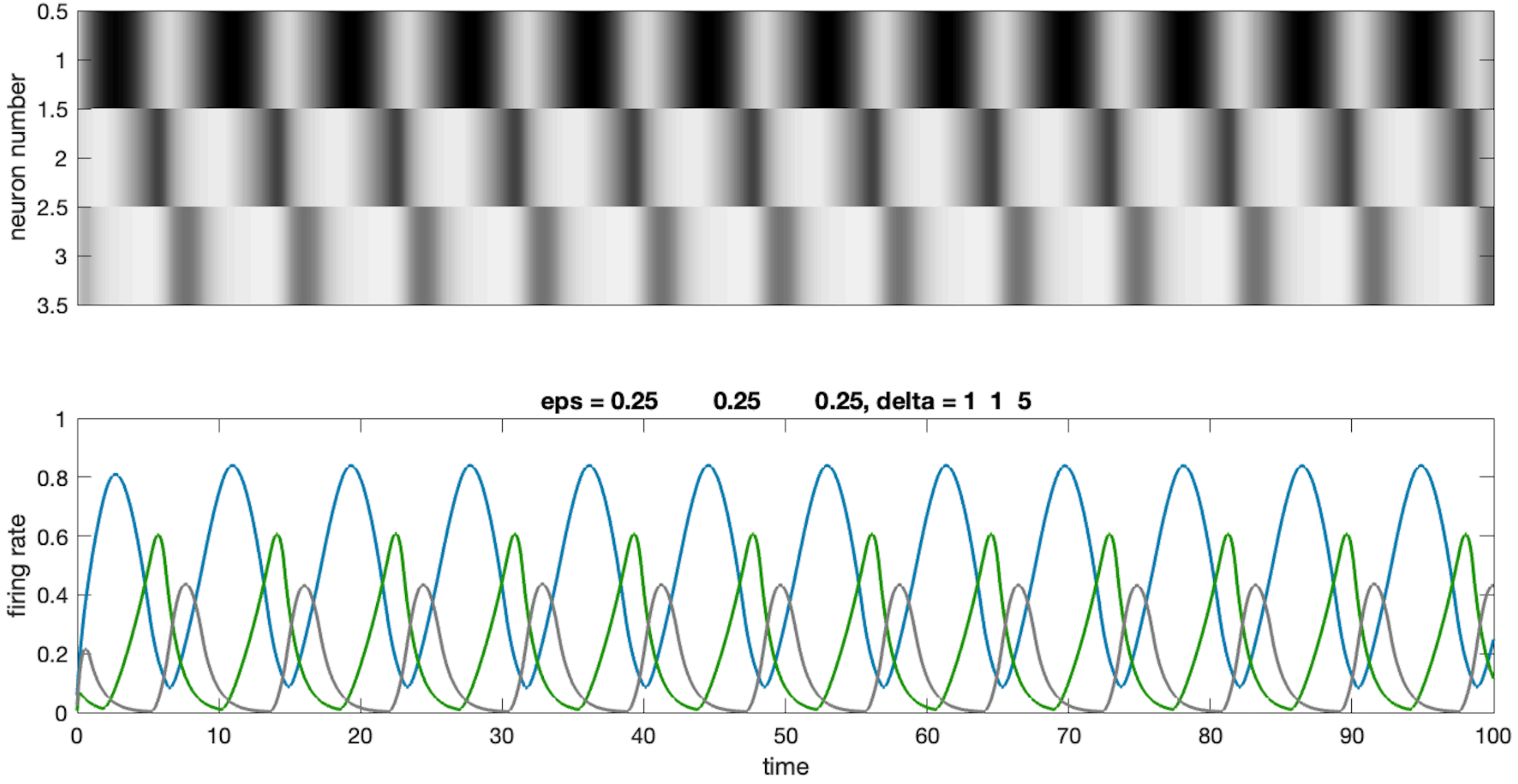
eps = 0.45 0.25 0.05, delta = 5 0.5 3



gCTLN #2

unstable fixed point: [0.2945 0.0858 0.3467]

eps = 0.25 0.25 0.25, delta = 1 1 5

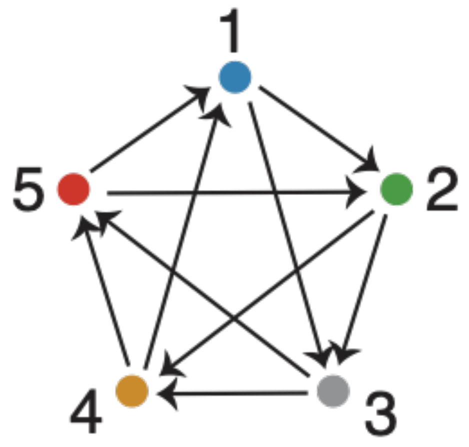


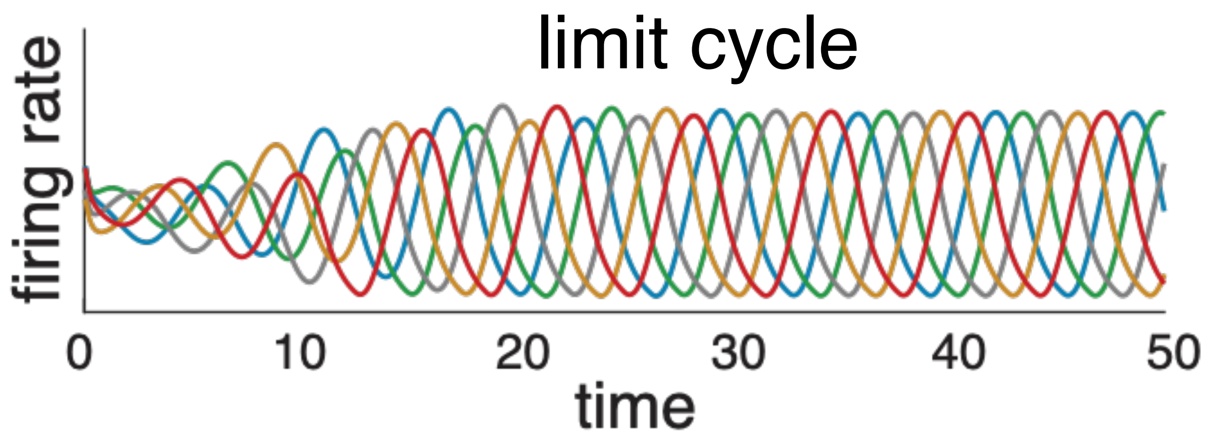
Fun examples!

$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + \theta\right]_+$$

Gaudí attractor

$\varepsilon = 0.1,\ \delta = 0.12,\ \theta = 1$





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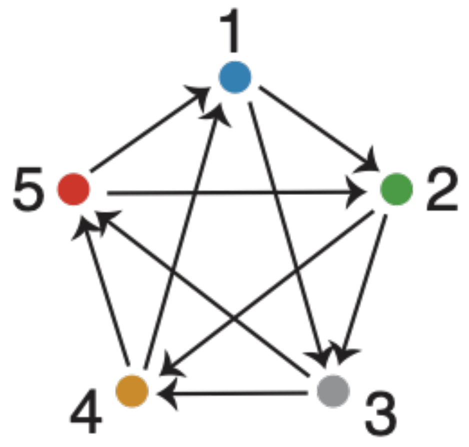
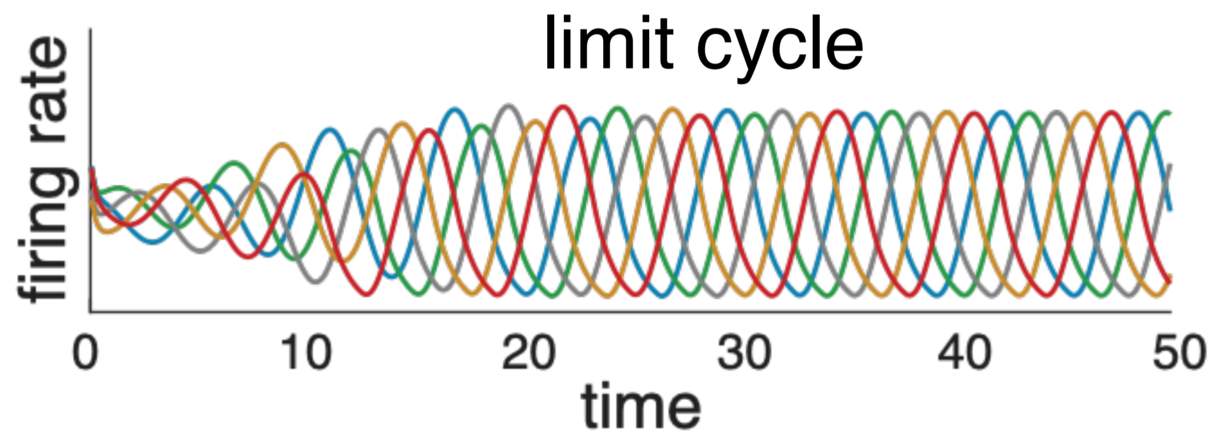
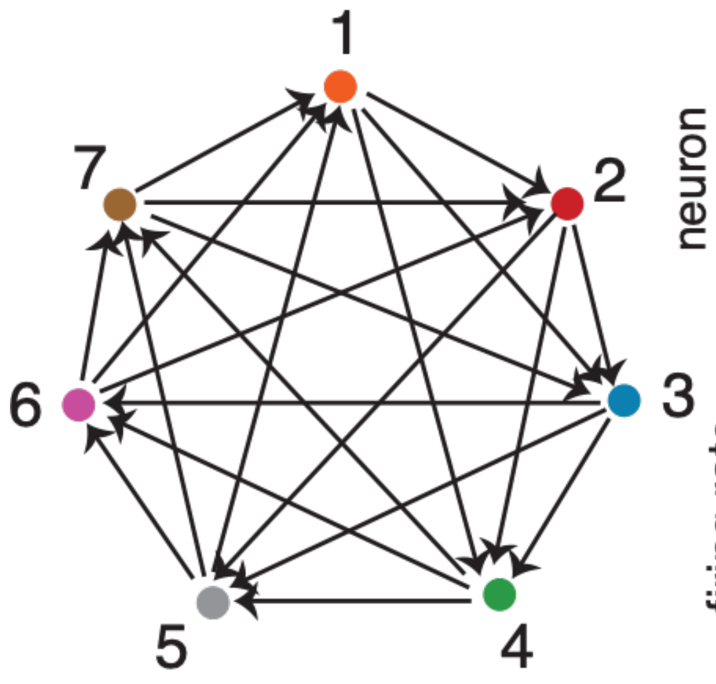
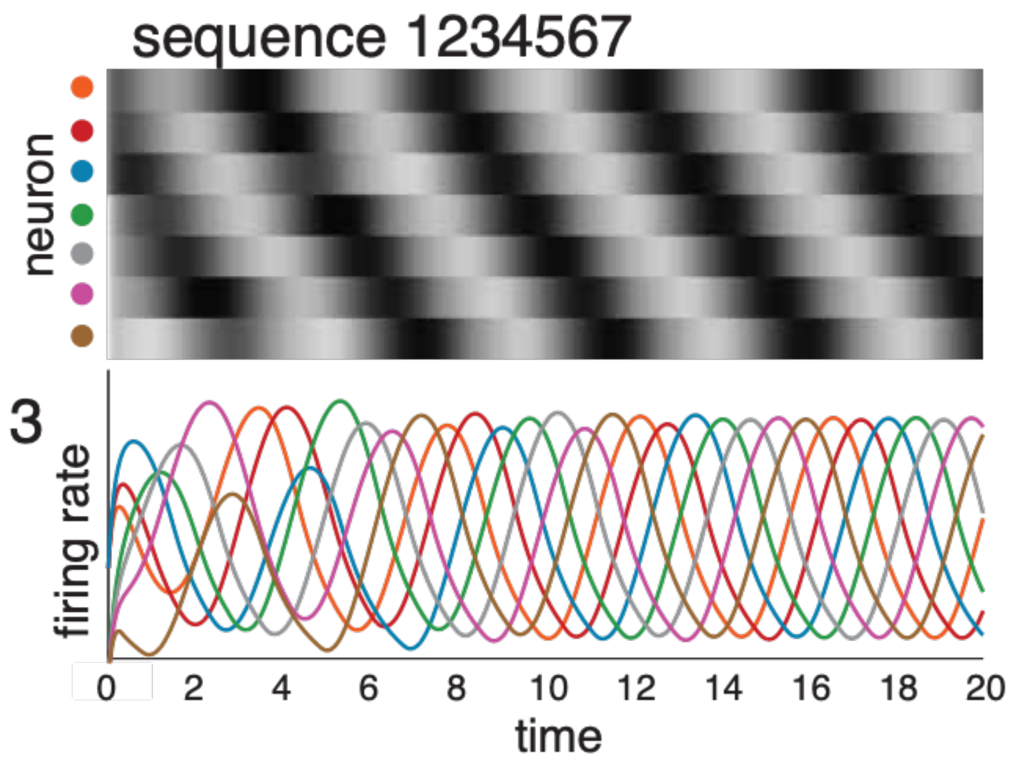




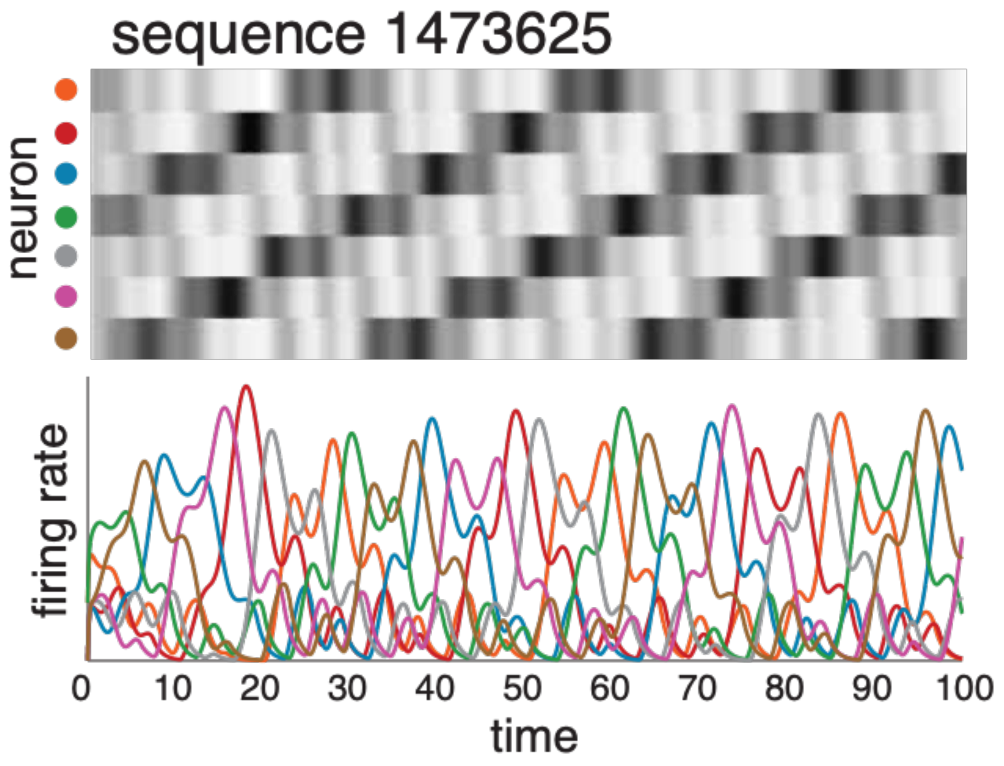
n = 7 star (another tournament)

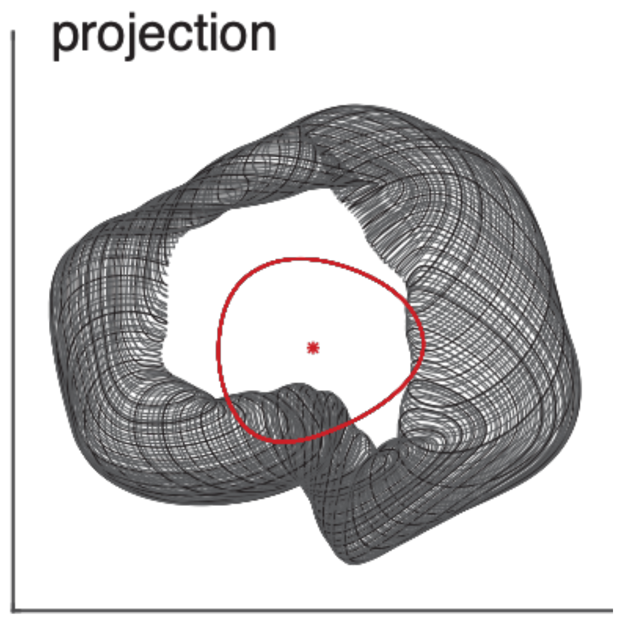


limit cycle



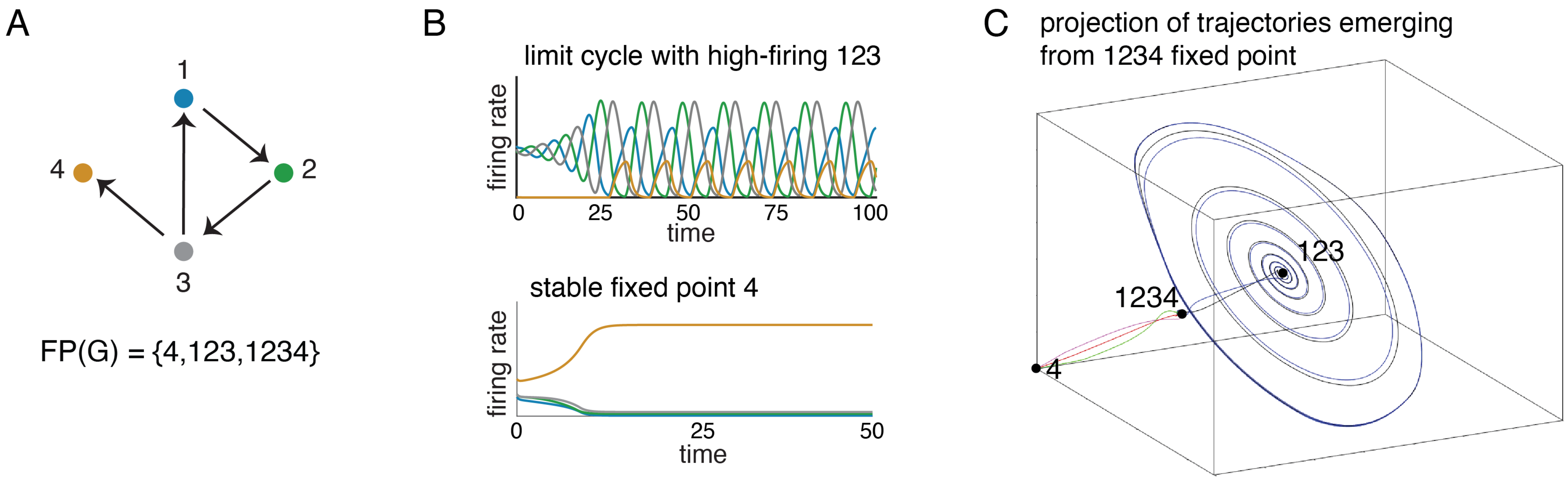
quasiperiodic attractor





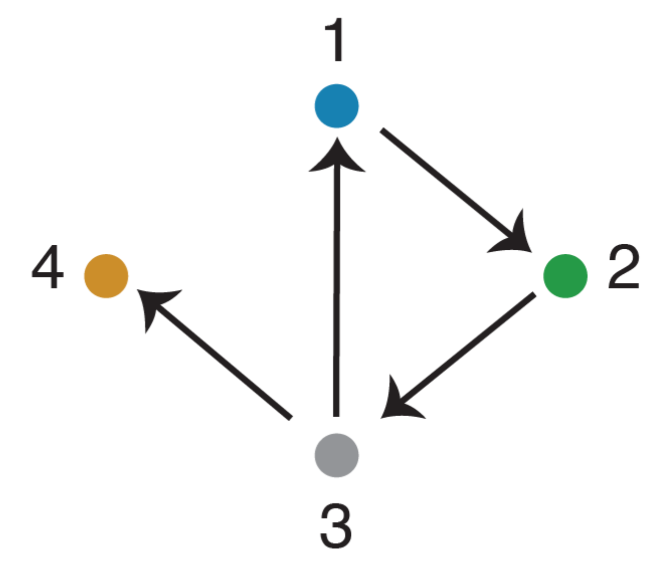
Diversity of emergent dynamics in competitive TLNs, Morrison, et. al., SIADS 2024

Fun examples!



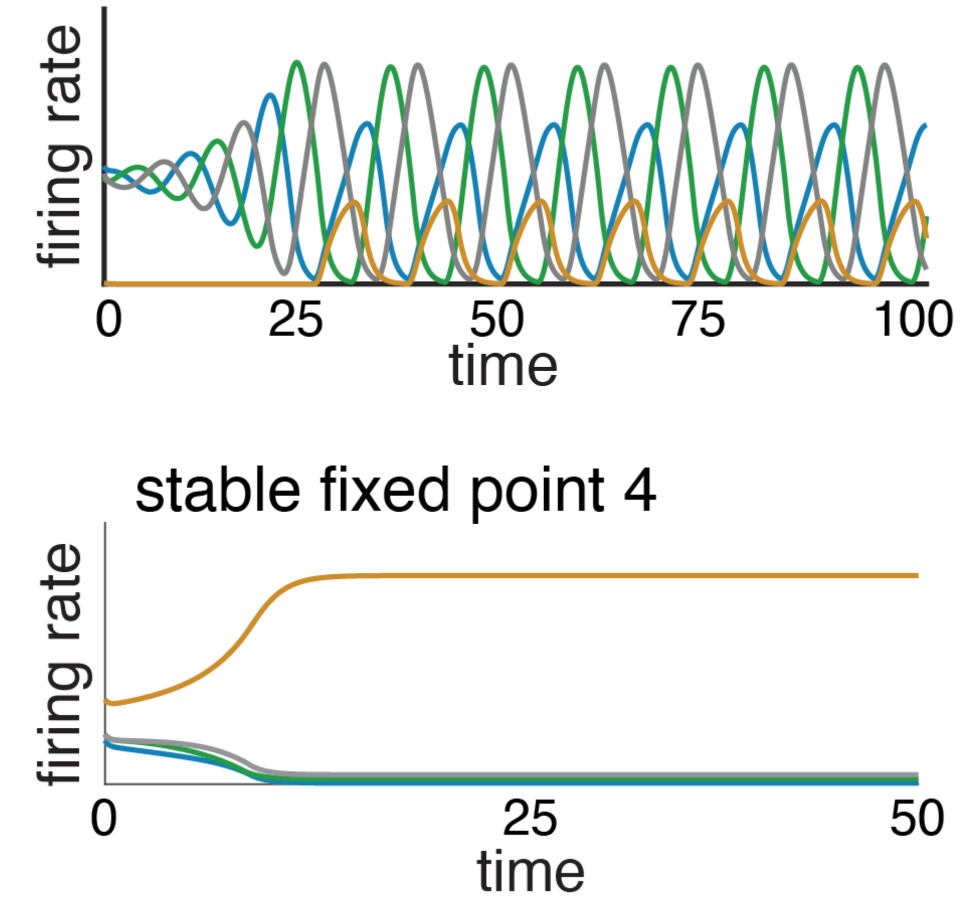
Fun examples!

A



FP(G) = {4,123,1234}

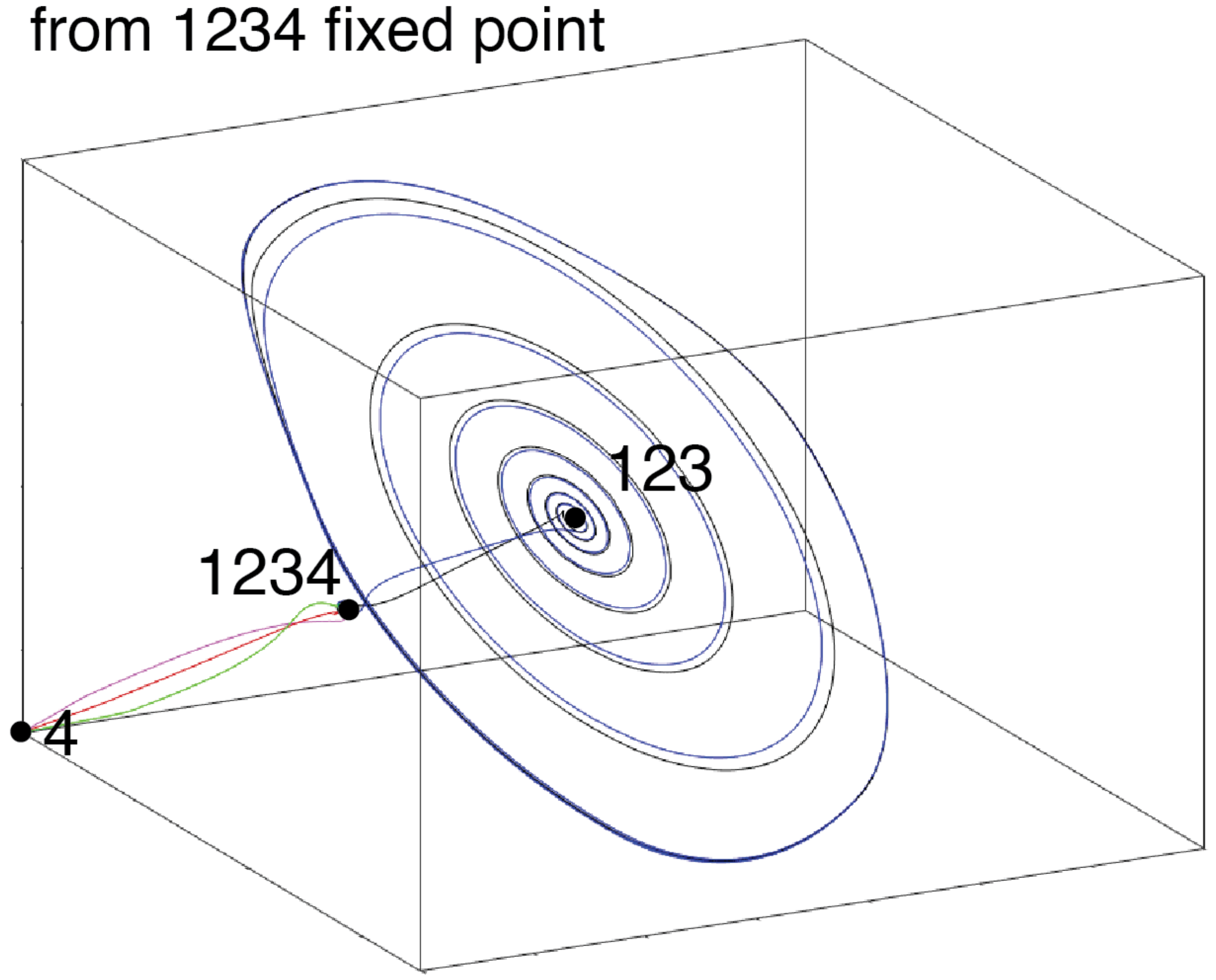
B


limit cycle with high-firing 123

stable fixed point 4

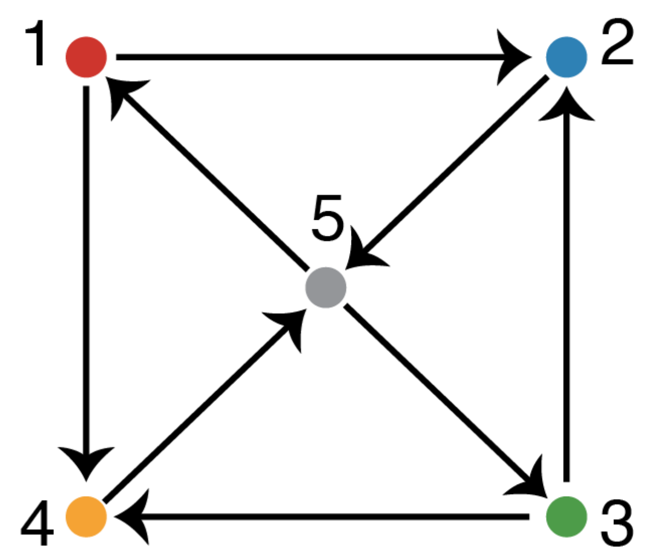
C

projection of trajectories emerging from 1234 fixed point



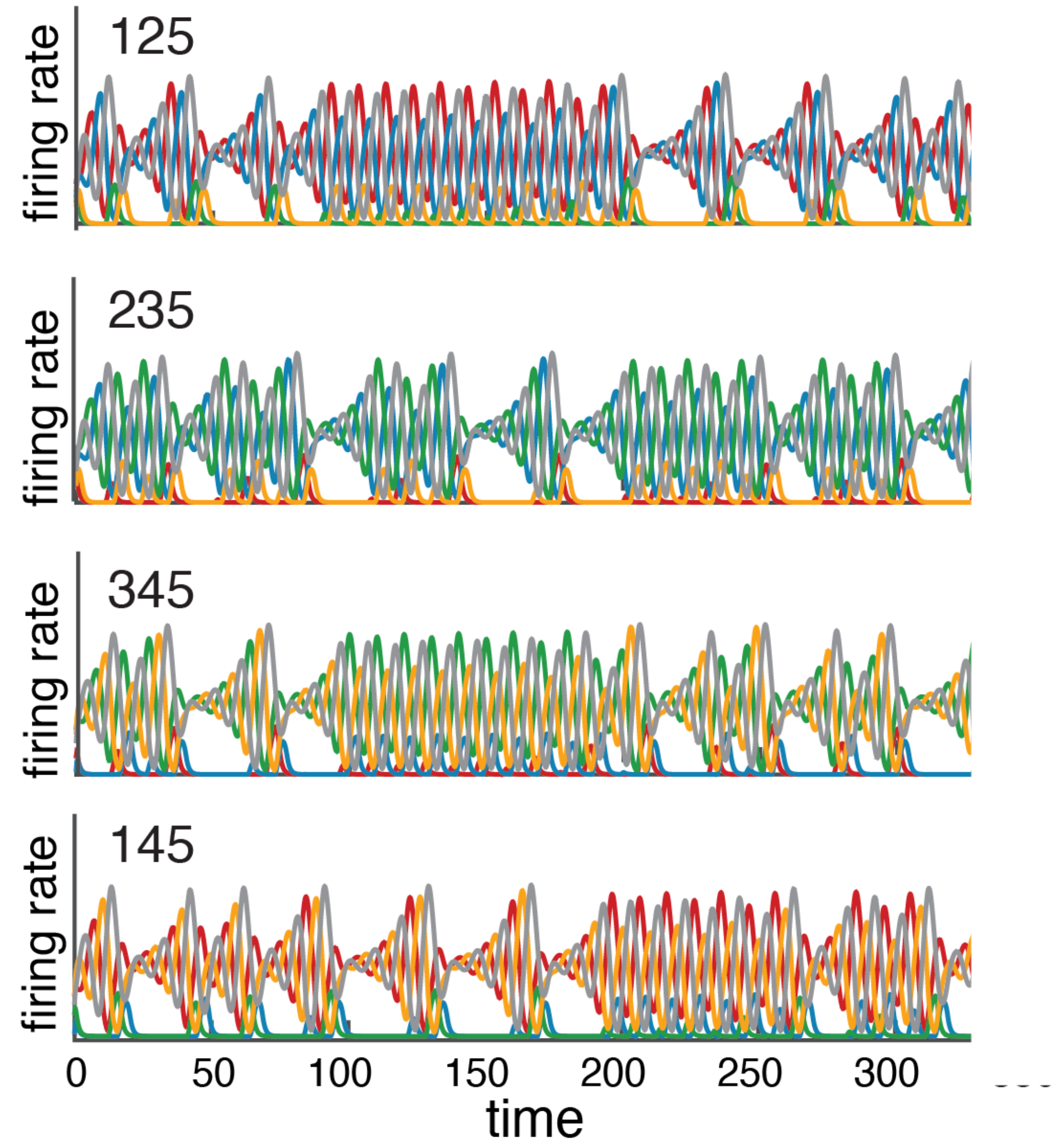
D

baby chaos



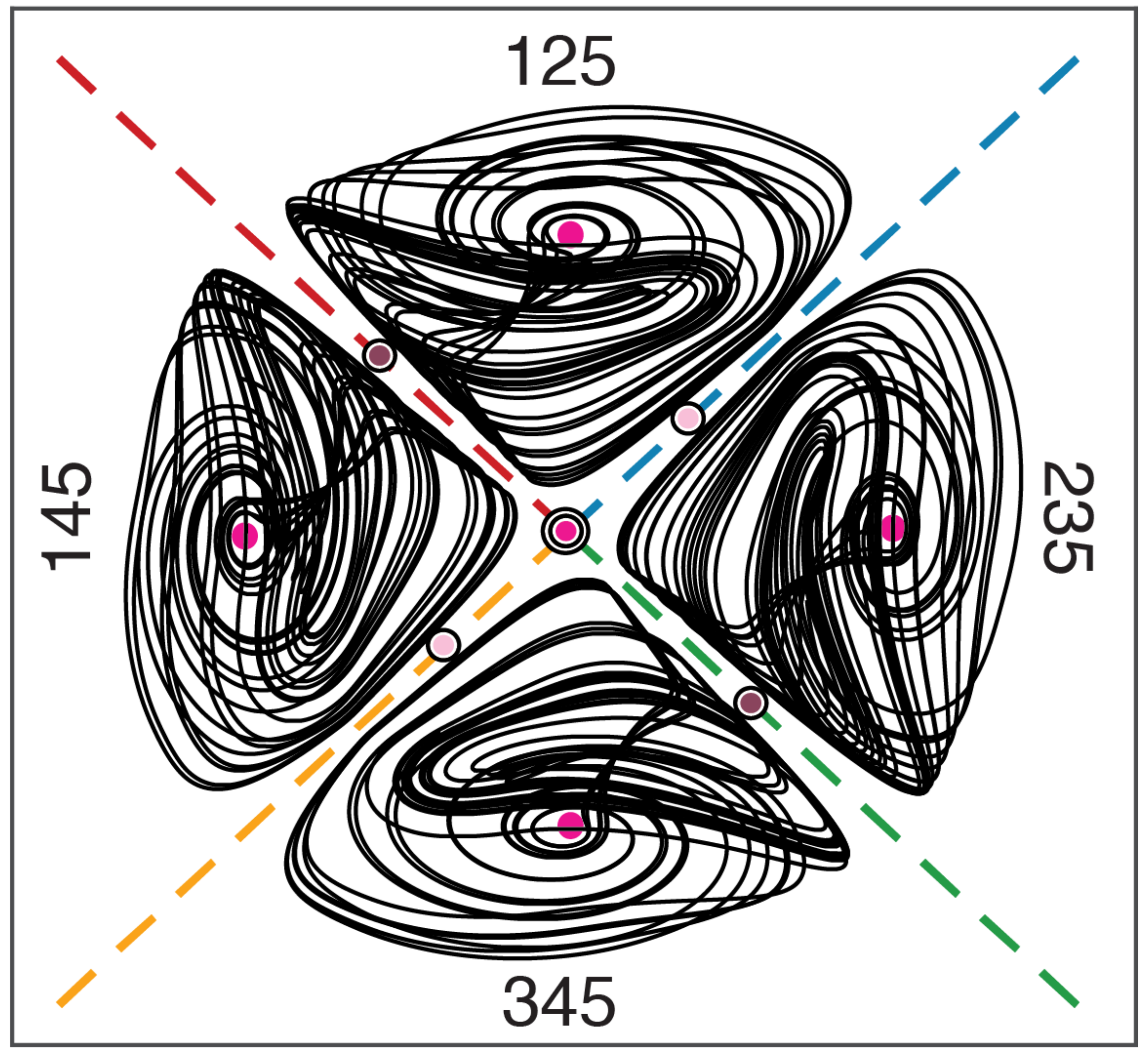
FP(G) = {125, 235, 345,145, 1235, 2345,1345,1245, 12345}

E



F

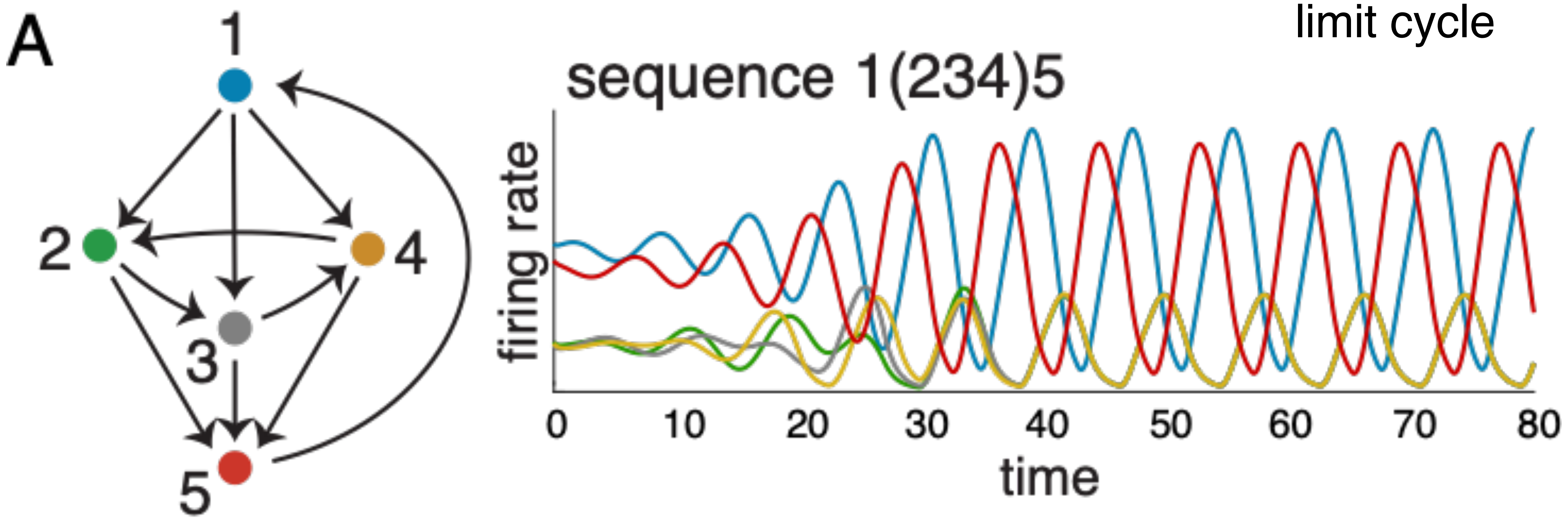
projection of all fixed points and attractors



Fun examples!

$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + \theta\right]_+$$

graph has an automorphism:
(234) symmetry
↓
234 synchrony



limit cycle

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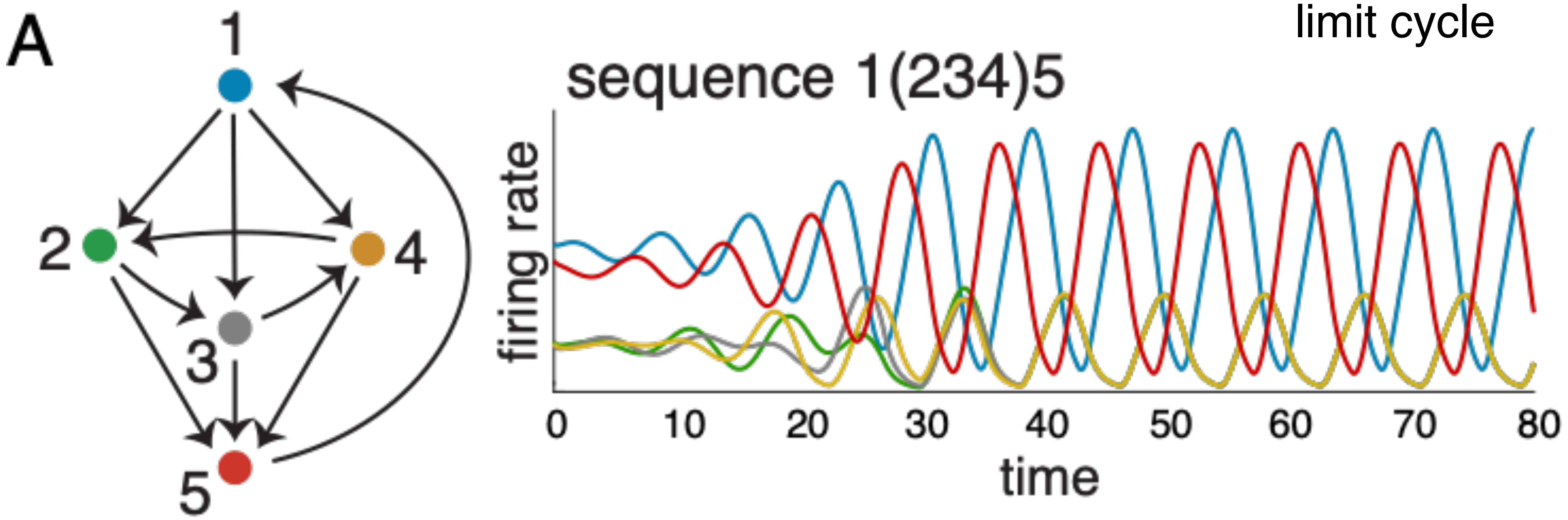
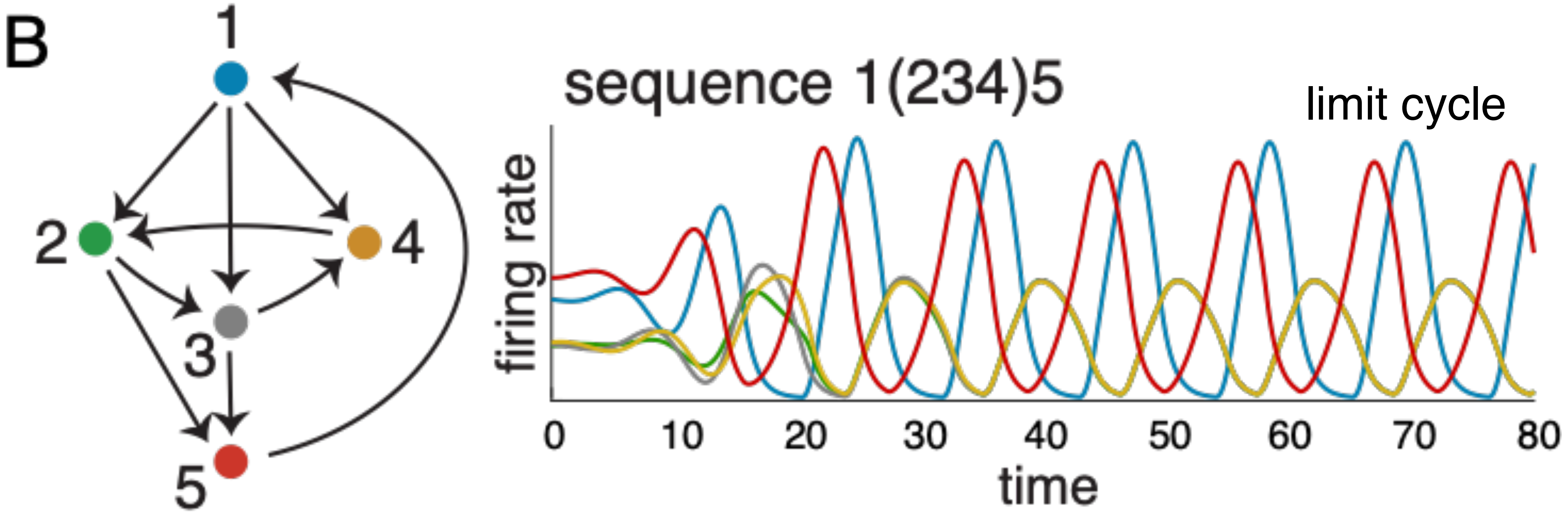


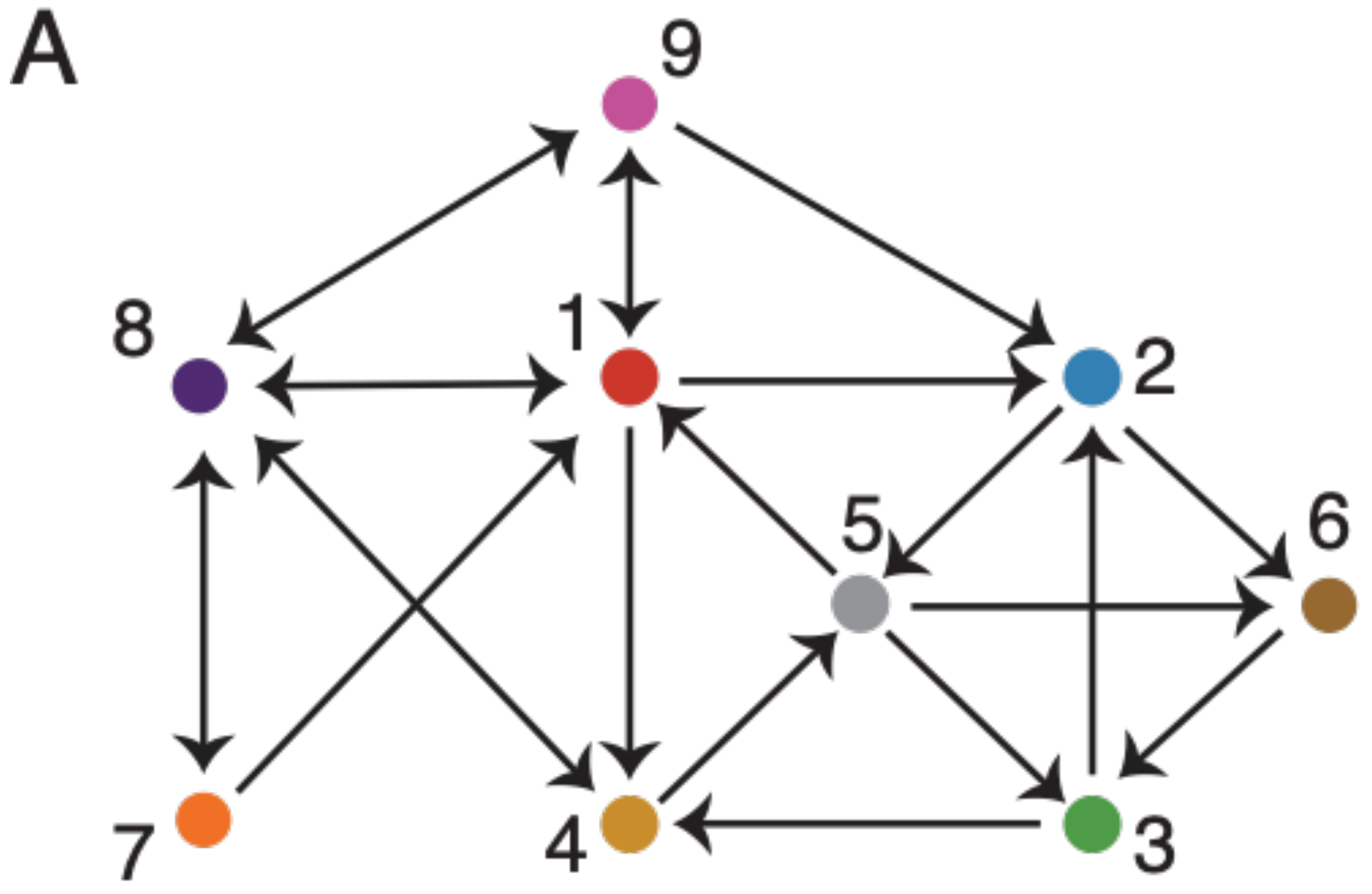
limit cycle

“surprise symmetry”
graph symmetry
is broken, but
234 synchrony persists!

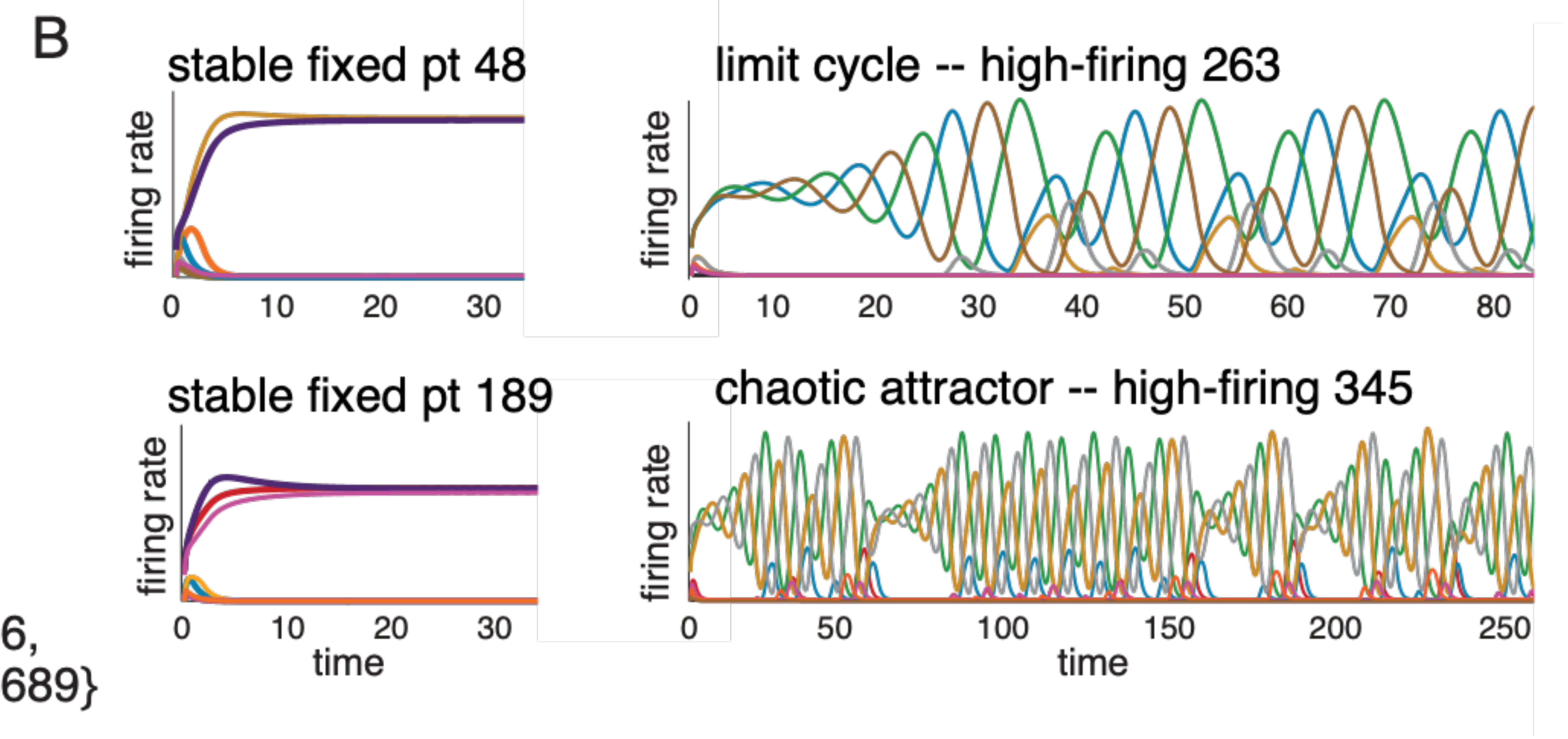


limit cycle

Coexistence of different attractors in (asymmetric) TLNs



FP(G) = {**48, 189, 236, 345,** 1289, 1345, 1489, 2345, 12345, 12458, 12689, 123456, 123689, 124568, 124589, 1234689, 1245689}



Different initial conditions yield different attractors,
but the equations of the network are identical in each case.

You can play with these yourself with simple code for CTLN simulations:

https://github.com/ccurto/CTLN-Basic-2.0

Matlab code was written to accompany this paper:
Diversity of emergent dynamics in competitive TLNs, SIADS 2024
https://arxiv.org/abs/1605.04463

A more general review paper:
C. Curto, K. Morrison. *Graph rules for recurrent network dynamics: extended version* (2023).
https://arxiv.org/abs/2301.12638
shorter version: Notices of the AMS, 2023

Plan of the talk

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- Dominoes and inhibitory control
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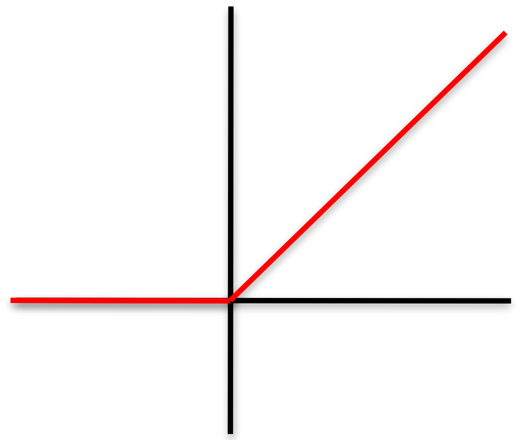
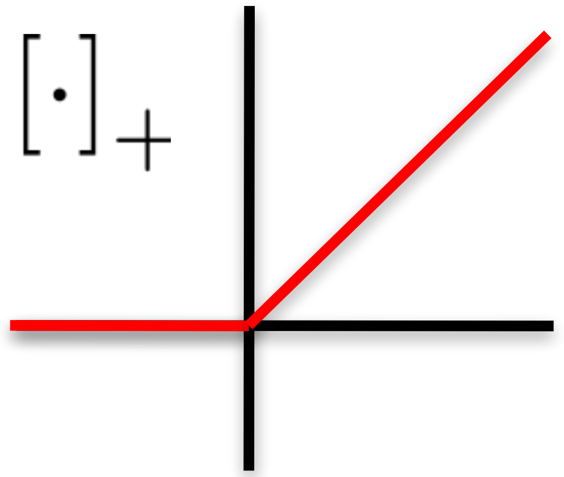
Why do we choose the threshold-linear (ReLU) for our nonlinearity?



TLNs as a patchwork of linear systems

$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + \theta\right]_+$$



Different linear system
of ODEs for each, indexed by:

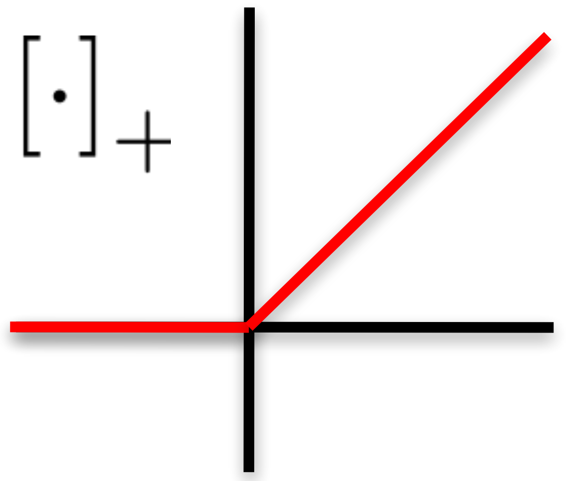
$$\sigma \subseteq [n]$$

$$\sigma = \{i \in [n] \mid y_i > 0\}$$

$$\begin{cases} \dfrac{dx_1}{dt} & = & -x_1 + \Big[\overbrace{\sum_{j=1}^{n} W_{1j}x_j + \theta}^{y_1}\Big]_+ \\ \dfrac{dx_2}{dt} & = & -x_2 + \Big[\underbrace{\sum_{j=1}^{n} W_{2j}x_j + \theta}_{y_2}\Big]_+ \\ & \vdots & \\ \dfrac{dx_n}{dt} & = & -x_n + \Big[\underbrace{\sum_{j=1}^{n} W_{nj}x_j + \theta}_{y_n}\Big]_+ \end{cases}$$

TLNs as a patchwork of linear systems

$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij} x_j + \theta \right]_+$$



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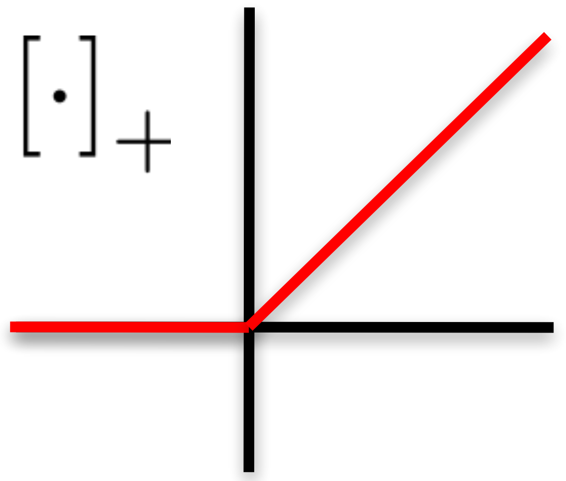
$$\boxed{\begin{array}{lll} \mathrm{FP}(W, b) & \overset{\text{def}}{=} & \{ \sigma \subseteq [n] \mid \sigma = \operatorname{supp} x^*, \text{ for some} \\ & & \text{fixed pt } x^* \text{ of the associated TLN} \} \end{array}}$$

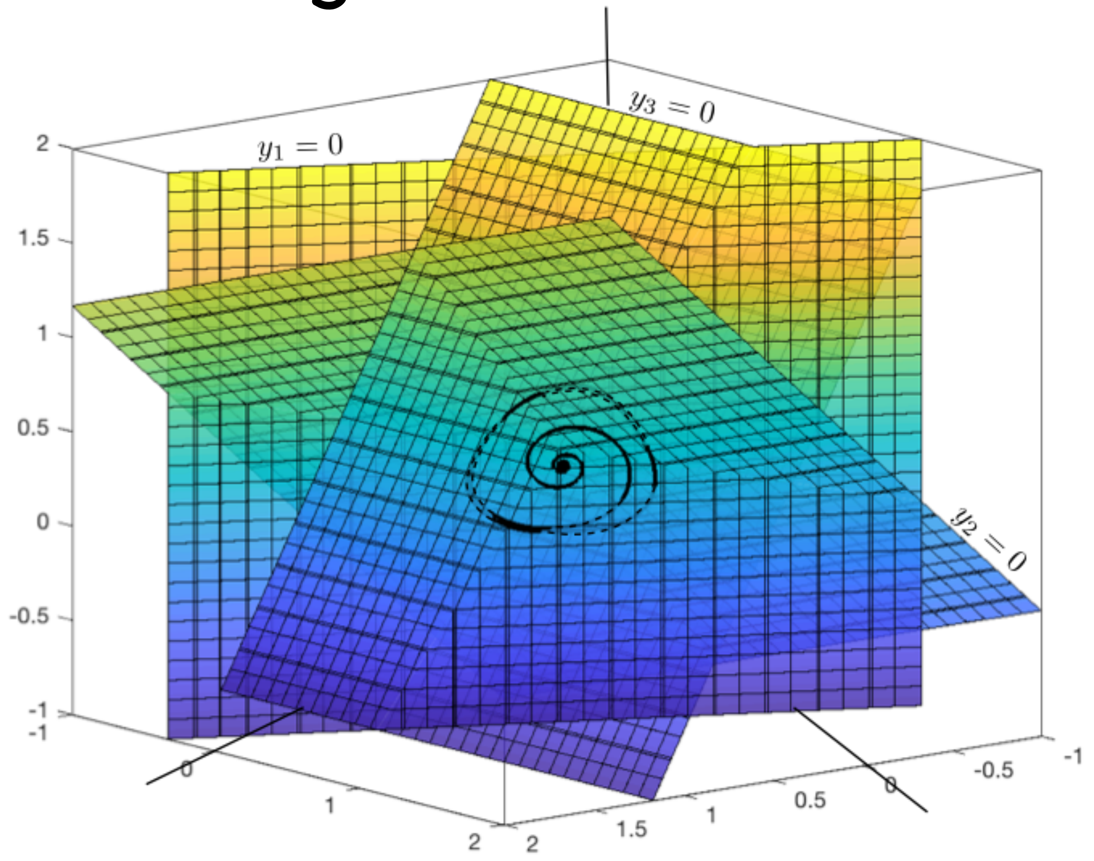
1-1 correspondence between fixed points and allowed supports

TLNs as a patchwork of linear systems

$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + \theta\right]_+$$



hyperplane arrangement defining linear chambers



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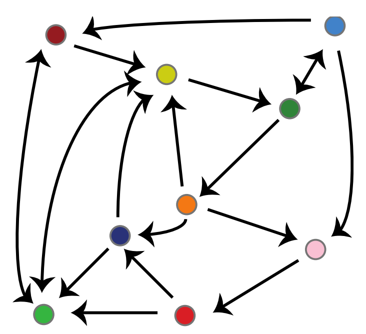
$$\boxed{\mathrm{FP}(W,b) \overset{\text{def}}{=} \{\sigma \subseteq [n] \mid \sigma = \operatorname{supp} x^*, \text{ for some fixed pt } x^* \text{ of the associated TLN}\}}$$

1-1 correspondence between fixed points and allowed supports

OLDER TECHNICAL RESULTS
for fixed points of TLNs

$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + \theta\right]_+$$



parity

Theorem 2.2 (parity [7]). *For any nondegenerate threshold-linear network* (W,b),

$$\sum_{\sigma \in \mathrm{FP}(W,b)} \mathrm{idx}(\sigma) = +1. \qquad \mathrm{idx}(\sigma) \overset{\text{def}}{=} \operatorname{sgn} \det(I - W_\sigma).$$

In particular, the total number of fixed points $|\mathrm{FP}(W,b)|$ *is always odd.*

Corollary 2.3. *The number of stable fixed points in a threshold-linear network of the form* (1.1) *is at most* 2^{n-1}.

sign conditions

Theorem 2.6. *Let* (W,b) *be a (non-degenerate) threshold-linear network with* $W_{ij} \leq 0$ *and* $b_i > 0$ *for all* $i,j \in [n]$. *For any nonempty* $\sigma \subseteq [n]$,

$$\sigma \in \mathrm{FP}(W,b) \quad \Leftrightarrow \quad \operatorname{sgn} s_i^\sigma = \operatorname{sgn} s_j^\sigma = -\operatorname{sgn} s_k^\sigma \text{ for all } i,j \in \sigma,\ k \notin \sigma. \qquad s_i^\sigma \overset{\text{def}}{=} \det((I - W_{\sigma\cup\{i\}})_i; b_{\sigma\cup\{i\}})$$

Moreover, if $\sigma \in \mathrm{FP}(W,b)$ *then* $\operatorname{sgn} s_i^\sigma = \operatorname{sgn}\det(I - W_\sigma) = \mathrm{idx}(\sigma)$ *for all* $i \in \sigma$.

domination

Theorem 2.11. *Let* (W,θ) *be a threshold-linear network. Then* $\sigma \in \mathrm{FP}(W,\theta)$ *if and only if the following two conditions hold:*

(i) σ *is domination-free, and*

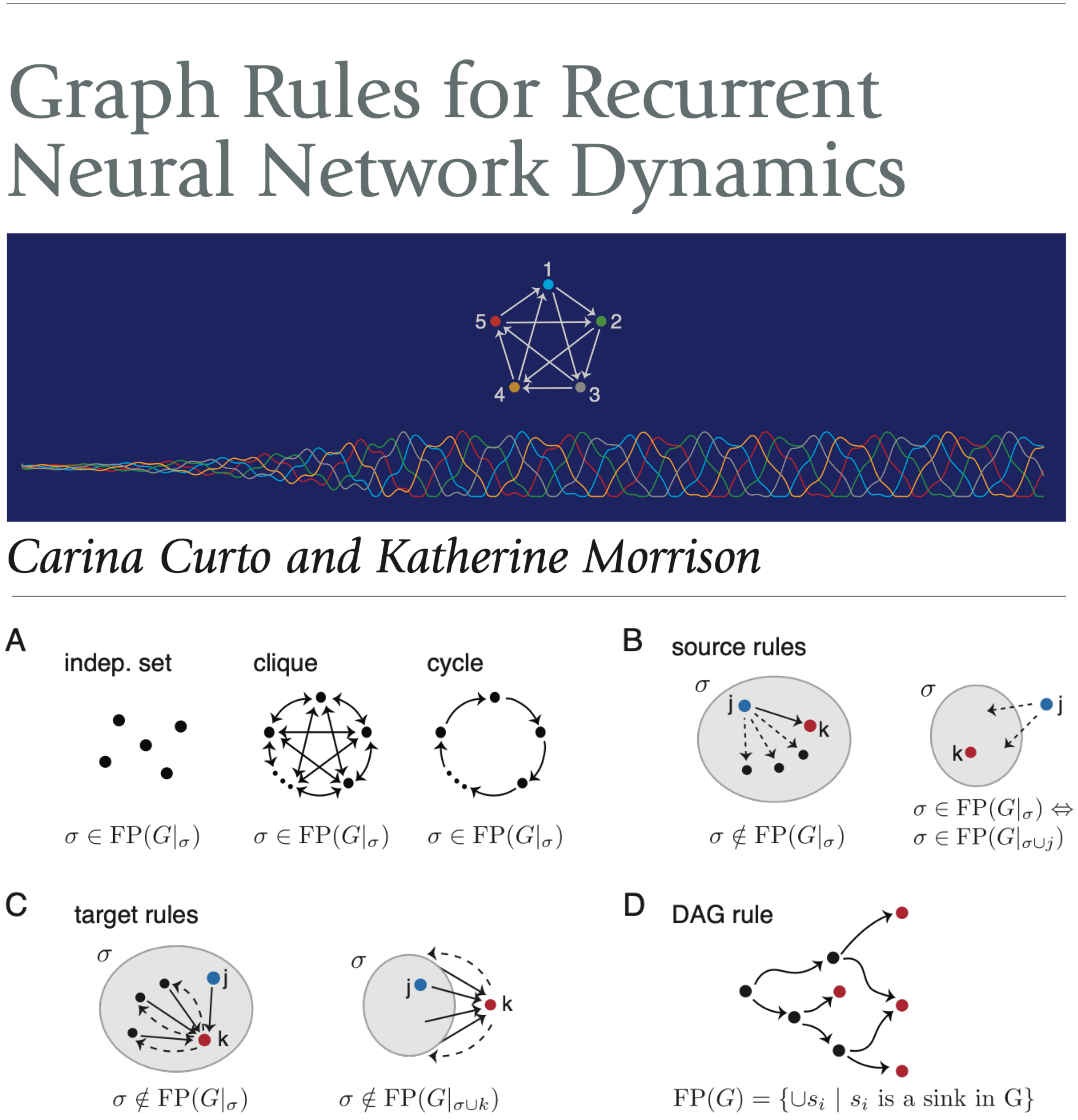
(ii) for each $k \notin \sigma$ *there exists* $j \in \sigma$ *such that* $j >_\sigma k$.

Diversity of emergent dynamics in competitive TLNs (SIADS, 2024)

Fixed points of threshold-linear networks. (Neural Computation, 2019)

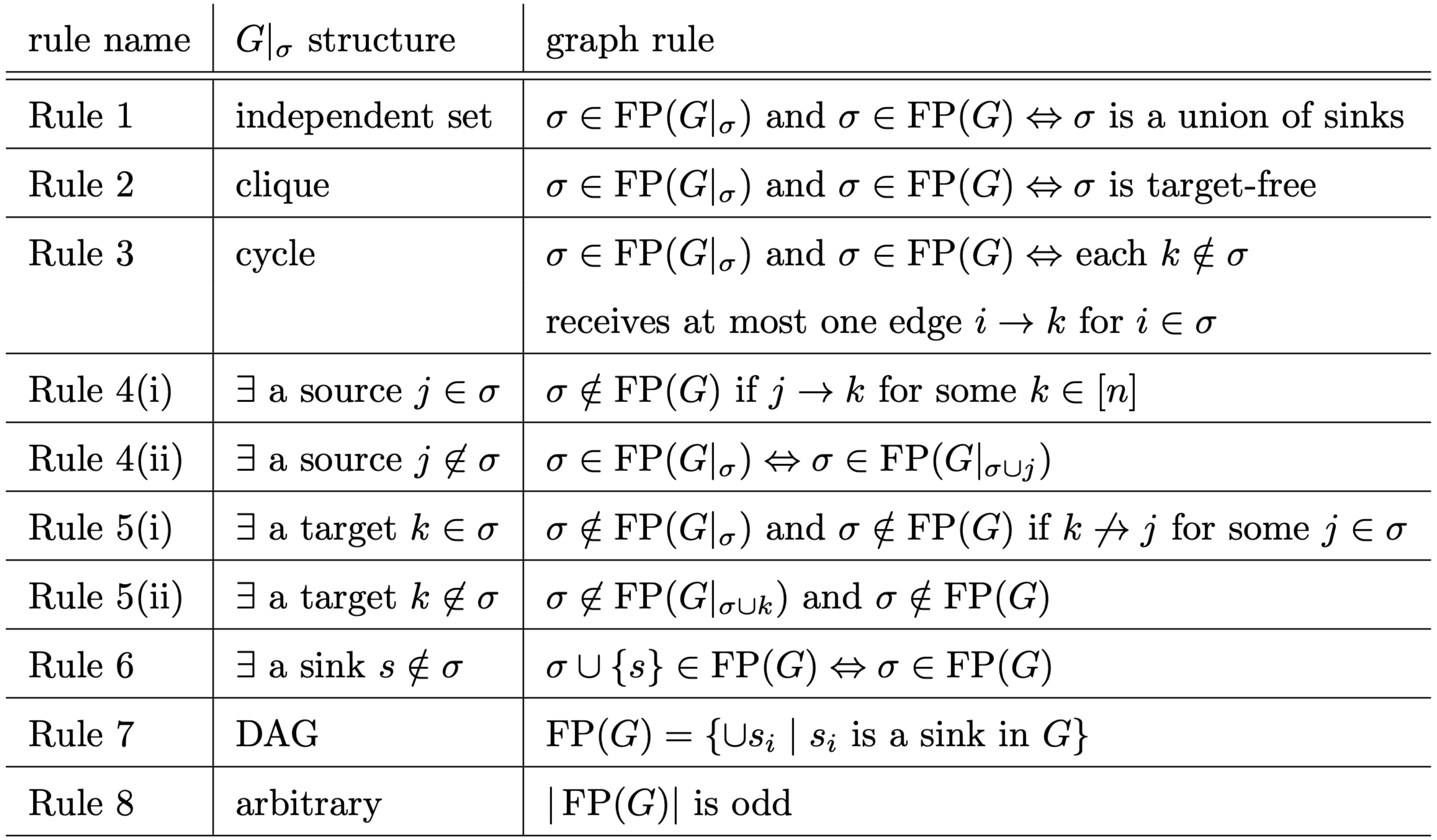
Graph rules for CTLN fixed point supports FP(G)

Notices of the AMS (2023)



Graph Rules for Recurrent Neural Network Dynamics

Carina Curto and Katherine Morrison

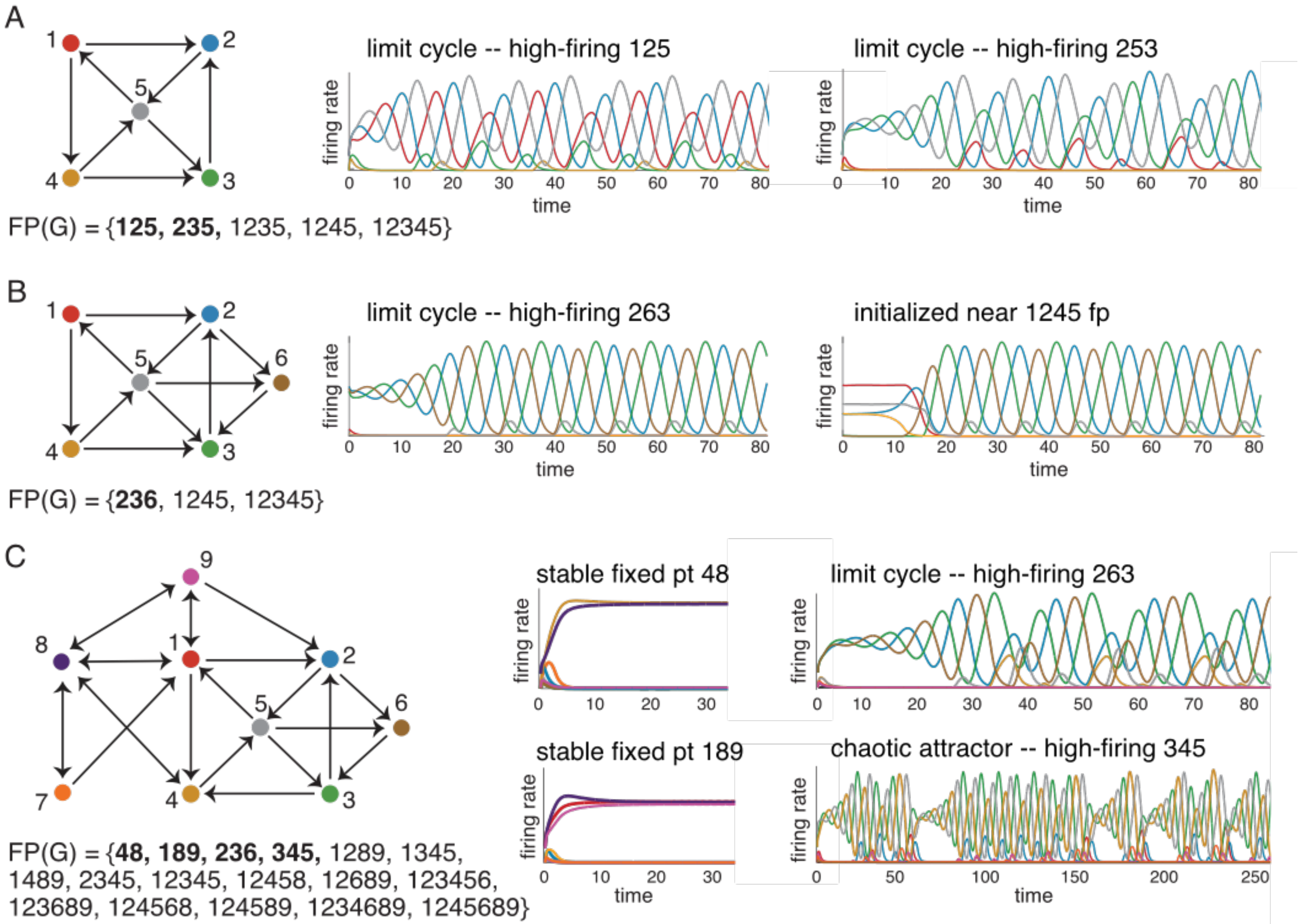


rule name	$G\vert_\sigma$ structure	graph rule
Rule 1	independent set	$\sigma \in \mathrm{FP}(G\vert_\sigma)$ and $\sigma \in \mathrm{FP}(G) \Leftrightarrow \sigma$ is a union of sinks
Rule 2	clique	$\sigma \in \mathrm{FP}(G\vert_\sigma)$ and $\sigma \in \mathrm{FP}(G) \Leftrightarrow \sigma$ is target-free
Rule 3	cycle	$\sigma \in \mathrm{FP}(G\vert_\sigma)$ and $\sigma \in \mathrm{FP}(G) \Leftrightarrow$ each $k \notin \sigma$ receives at most one edge $i \to k$ for $i \in \sigma$
Rule 4(i)	$\exists$ a source $j \in \sigma$	$\sigma \notin \mathrm{FP}(G)$ if $j \to k$ for some $k \in [n]$
Rule 4(ii)	$\exists$ a source $j \notin \sigma$	$\sigma \in \mathrm{FP}(G\vert_\sigma) \Leftrightarrow \sigma \in \mathrm{FP}(G\vert_{\sigma \cup j})$
Rule 5(i)	$\exists$ a target $k \in \sigma$	$\sigma \notin \mathrm{FP}(G\vert_\sigma)$ and $\sigma \notin \mathrm{FP}(G)$ if $k \not\to j$ for some $j \in \sigma$
Rule 5(ii)	$\exists$ a target $k \notin \sigma$	$\sigma \notin \mathrm{FP}(G\vert_{\sigma \cup k})$ and $\sigma \notin \mathrm{FP}(G)$
Rule 6	$\exists$ a sink $s \notin \sigma$	$\sigma \cup \{s\} \in \mathrm{FP}(G) \Leftrightarrow \sigma \in \mathrm{FP}(G)$
Rule 7	DAG	$\mathrm{FP}(G) = \{\cup s_i \mid s_i$ is a sink in $G\}$
Rule 8	arbitrary	$\vert\mathrm{FP}(G)\vert$ is odd

Table 1: Summary of derived graph rules.

C. Curto, K. Morrison. *Graph rules for recurrent network dynamics: extended version* (2023). https://arxiv.org/abs/2301.12638

Minimal fixed points give rise to attractors



FP(G) = {**125, 235,** 1235, 1245, 12345}

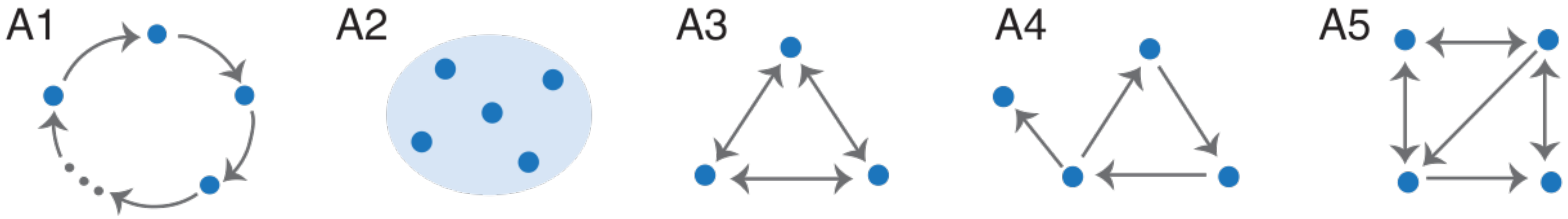
FP(G) = {**236**, 1245, 12345}

FP(G) = {**48, 189, 236, 345,** 1289, 1345, 1489, 2345, 12345, 12458, 12689, 123456, 123689, 124568, 124589, 1234689, 1245689}

Theorem: uniform in-degree

(yields Rules 1-3)

G has uniform in-degree if all nodes have the same in-degree d.



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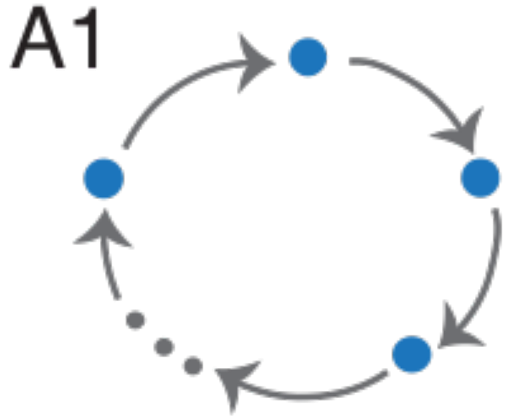
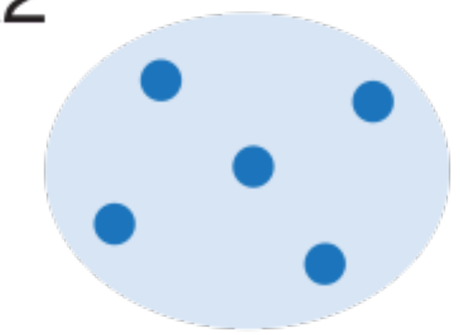
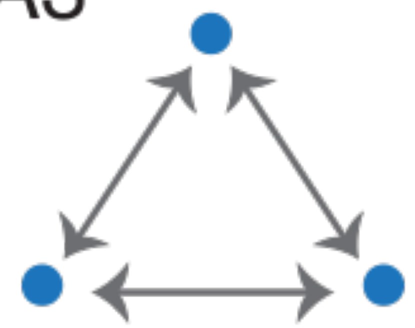
A1



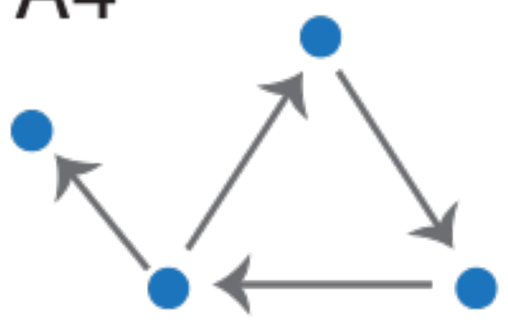
A2



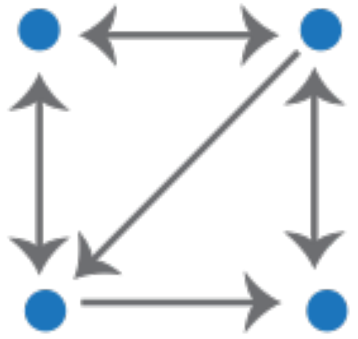
A3



A4



A5



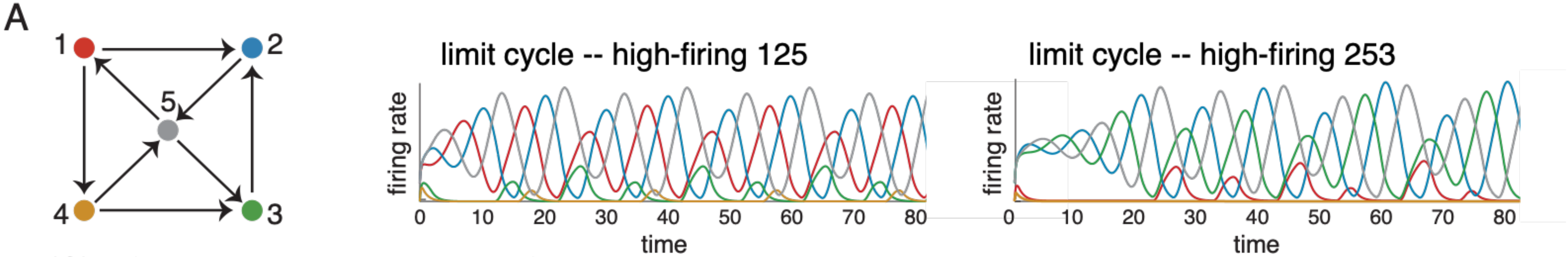
Theorem. Let $G|_\sigma$ be an induced subgraph of uniform in-degree d. Then

$\sigma \in \mathrm{FP}(G)$ $\Leftrightarrow$ no node outside $G|_\sigma$ receives d+1 (or more) edges from σ

Which cycles have surviving fixed points?

Corollary 1. Let G be a cycle.

Fixed point survives $\Longleftrightarrow$ no node outside G receives 2 (or more) edges from G


limit cycle -- high-firing 125

limit cycle -- high-firing 253

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Which cycles have surviving fixed points?

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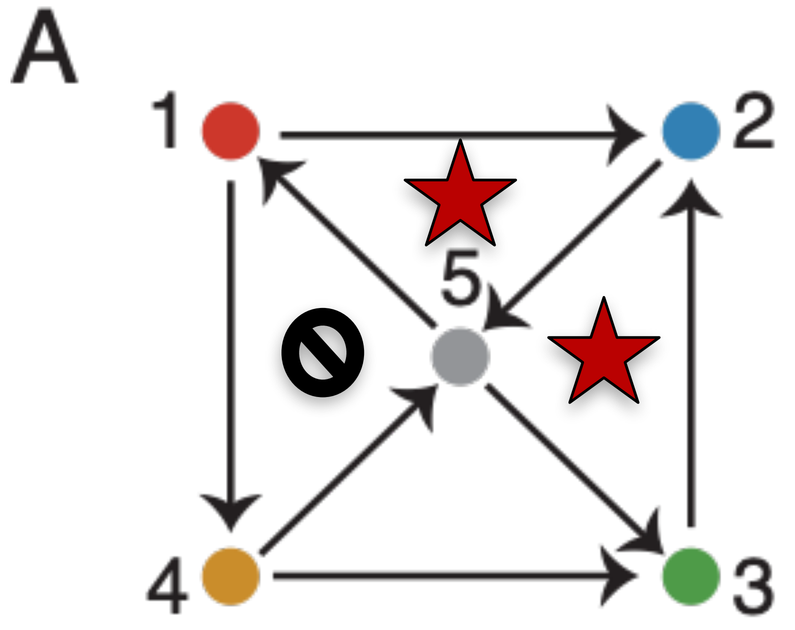
Fixed point survives ⟺ no node outside G receives 2 (or more) edges from G



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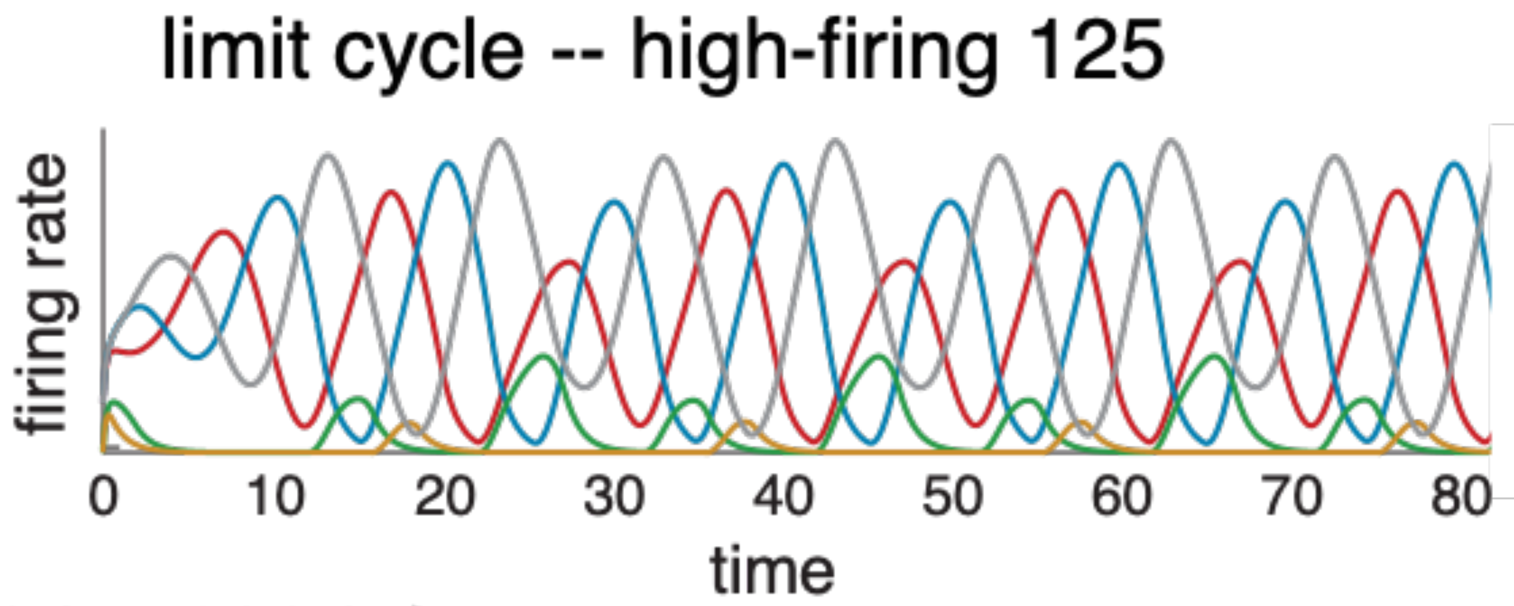
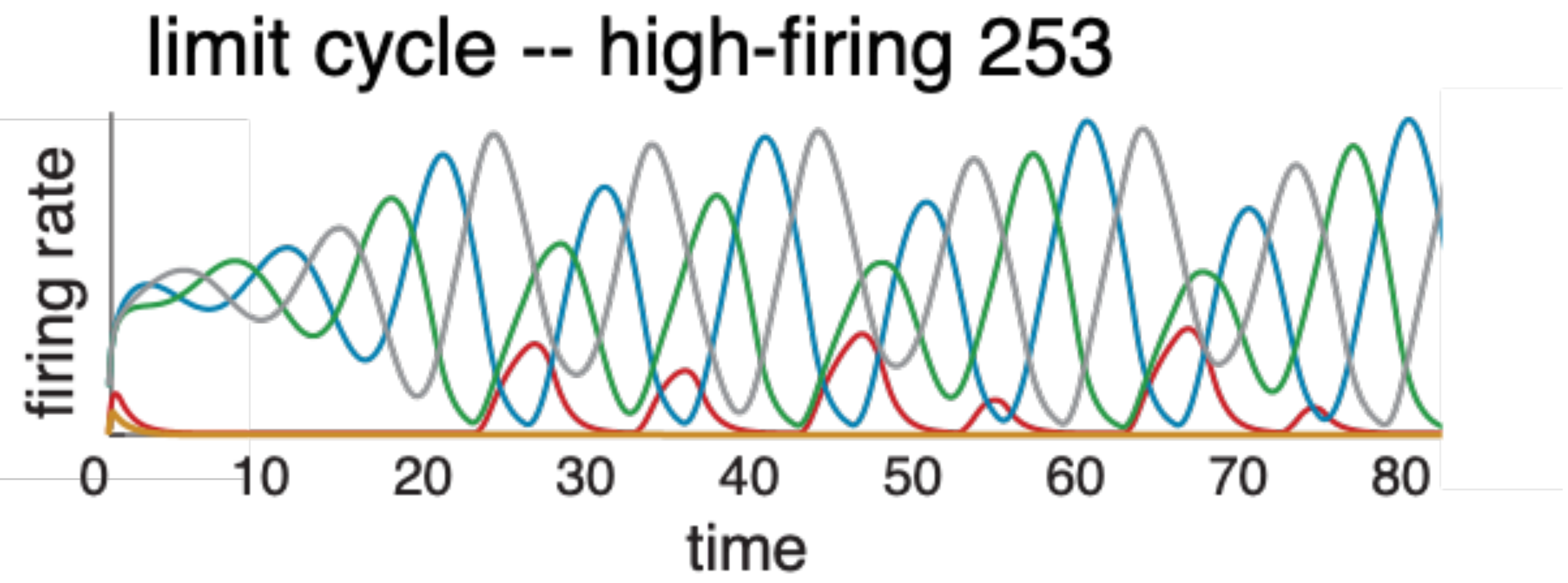




Observations about competitive TLNs

$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + b_i\right]_+$$

1. Directed graphs (non-symmetric W) is necessary to get dynamic attractors that (as opposed to fixed points).

2. Unstable fixed points matter — b/c of the Perron-Frobenius theorem.

3. Degeneracy: attractors can be preserved with changing weights (selectively).

4. Architecture provides serious constraints, not everything is possible!

5. The same in/out-degree distribution can correspond to networks with wildly different dynamics.

6. Sequences emerge very naturally because of the inhibition. There is no need for a synaptic chain in the architecture.

recent survey if you want to know more:

Curto & Morrison, Notices of the AMS, 2023

Plan of the talk

- Brief intro to TLNs, CTLNs, and gCTLNs
- Fixed points and attractors and graph rules
- Domination
- Dominoes and inhibitory control
- E-I TLNs
- Domination-reduction in connectomes

Focus on one very important graph property:

domination

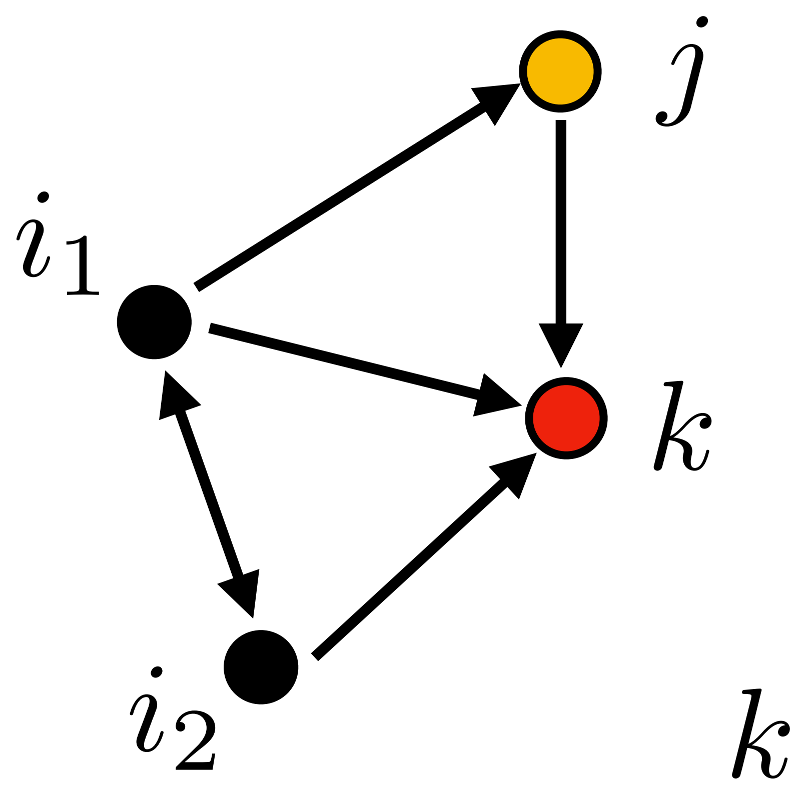
Domination

Definition 1.1. Let $j, k \in [n]$ be vertices of G. We say that k *graphically dominates* j in G if the following two conditions hold:

(i) For each vertex $i \in [n] \setminus \{j, k\}$, if $i \to j$ then $i \to k$.

(ii) $j \to k$ and $k \not\to j$.

If there exists a k that graphically dominates j, we say that j is a *dominated node* (or *dominated vertex*) of G. If G has no dominated nodes, we say that it is *domination free*.



$k > j$

"k dominates j"

"j is a dominated node"

Curto, Geneson, Morrison, 2019

domination is a property of G

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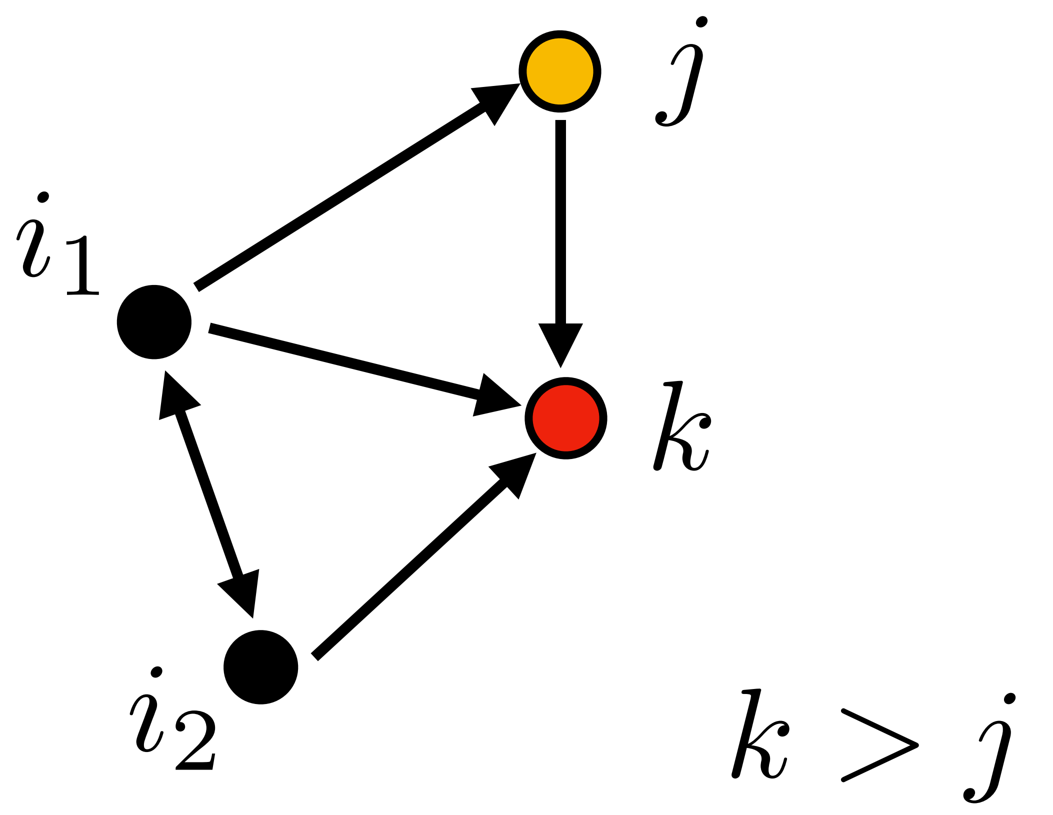

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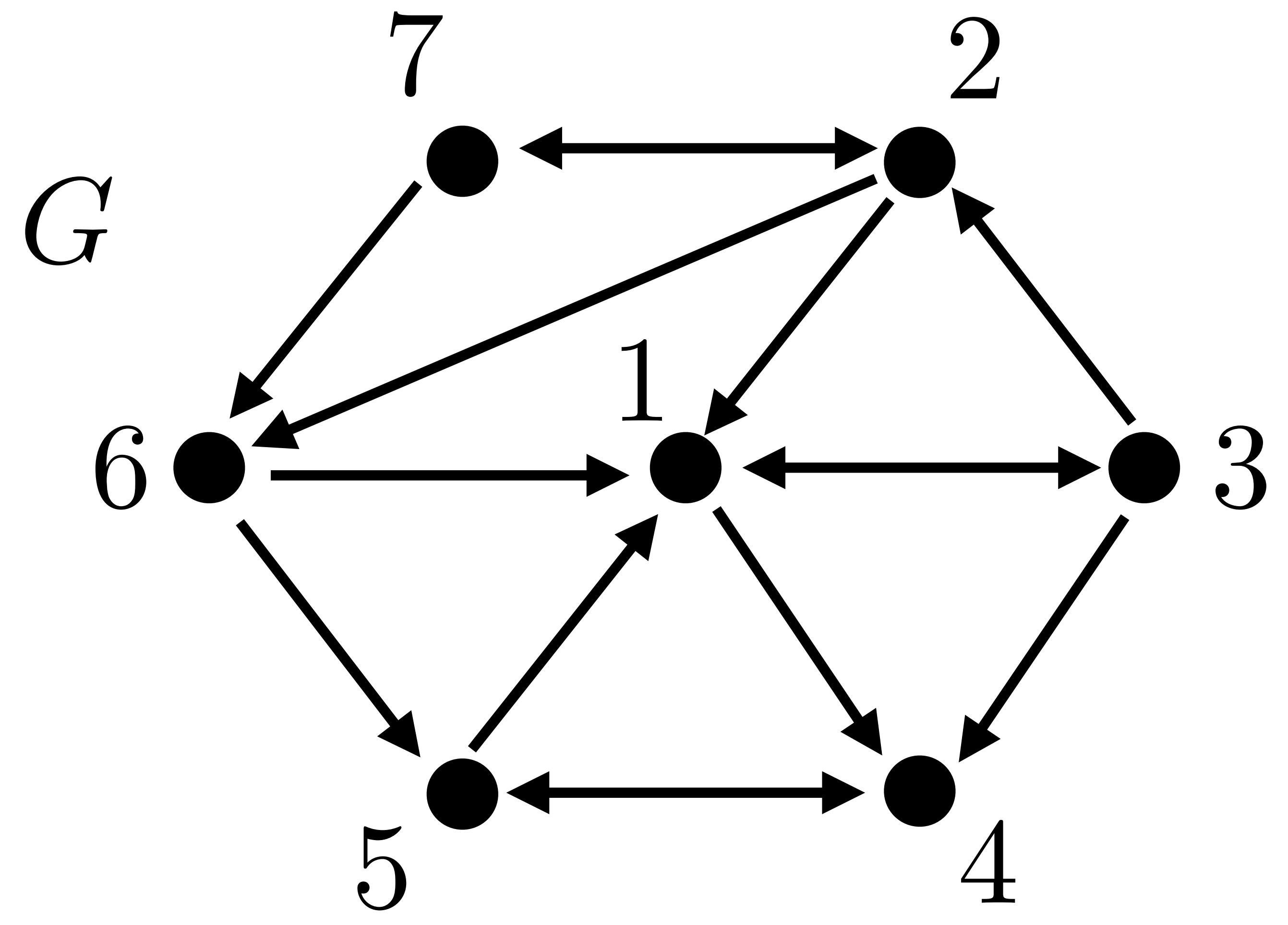
Example



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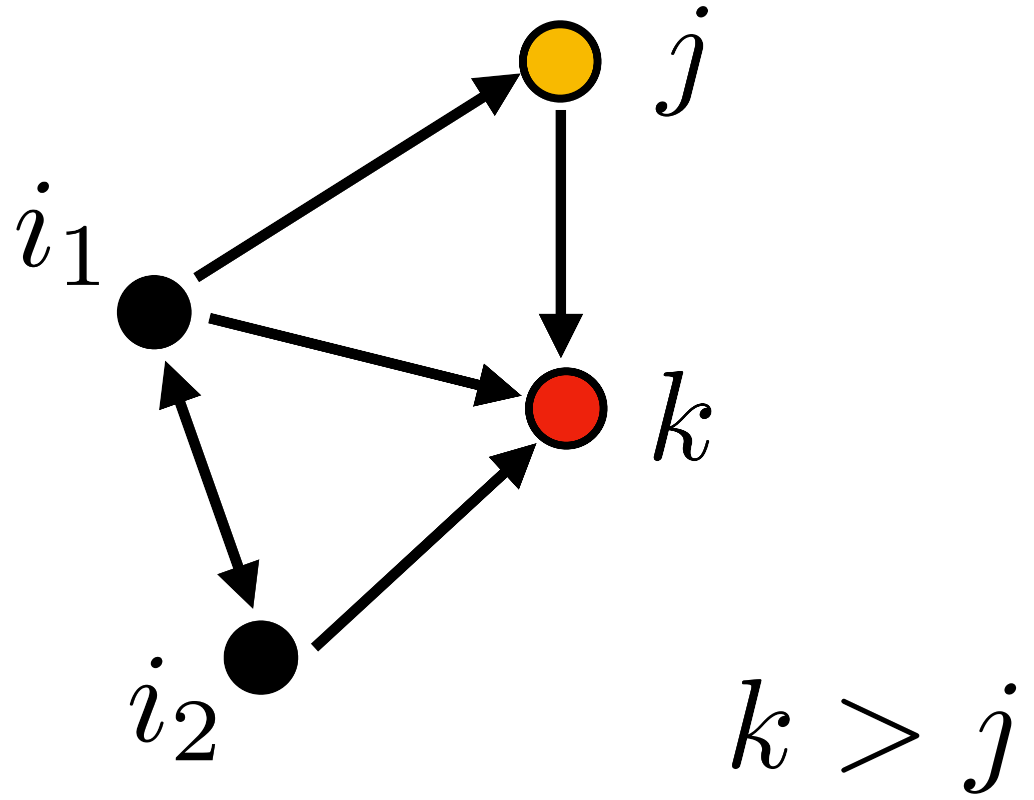

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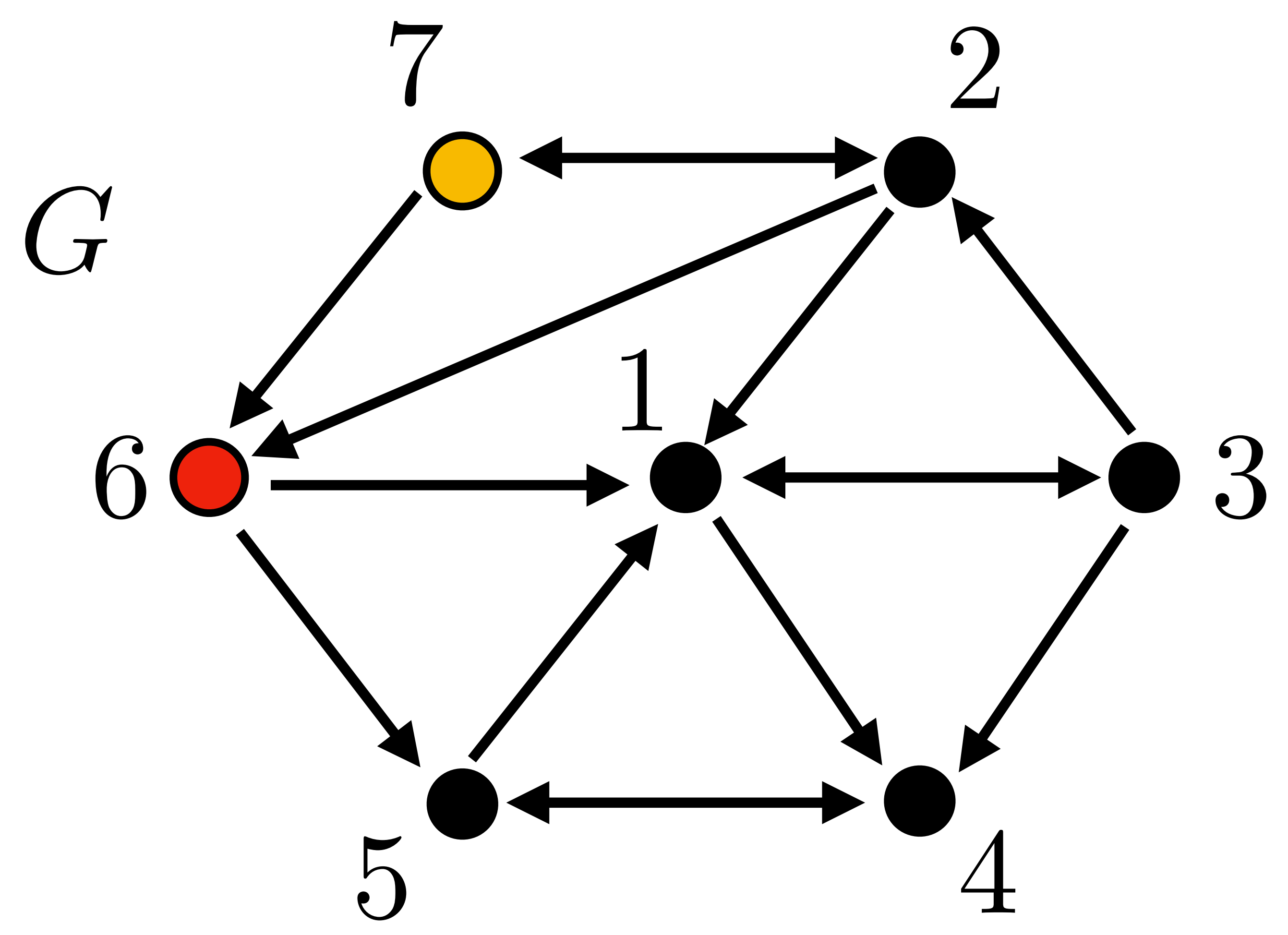

$6 > 7$

domination is a property of G

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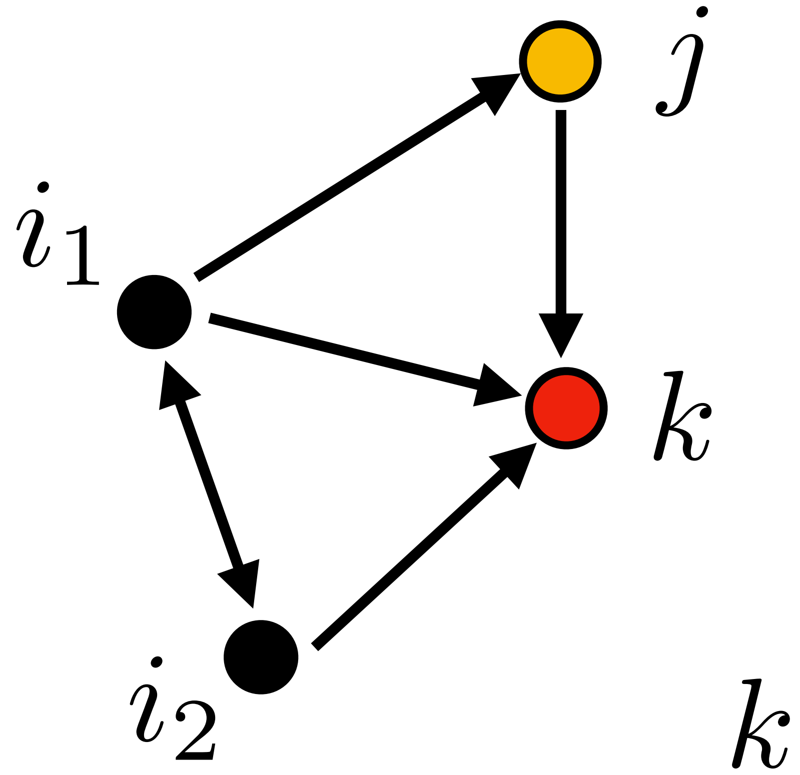

$k > j$

"k dominates j"

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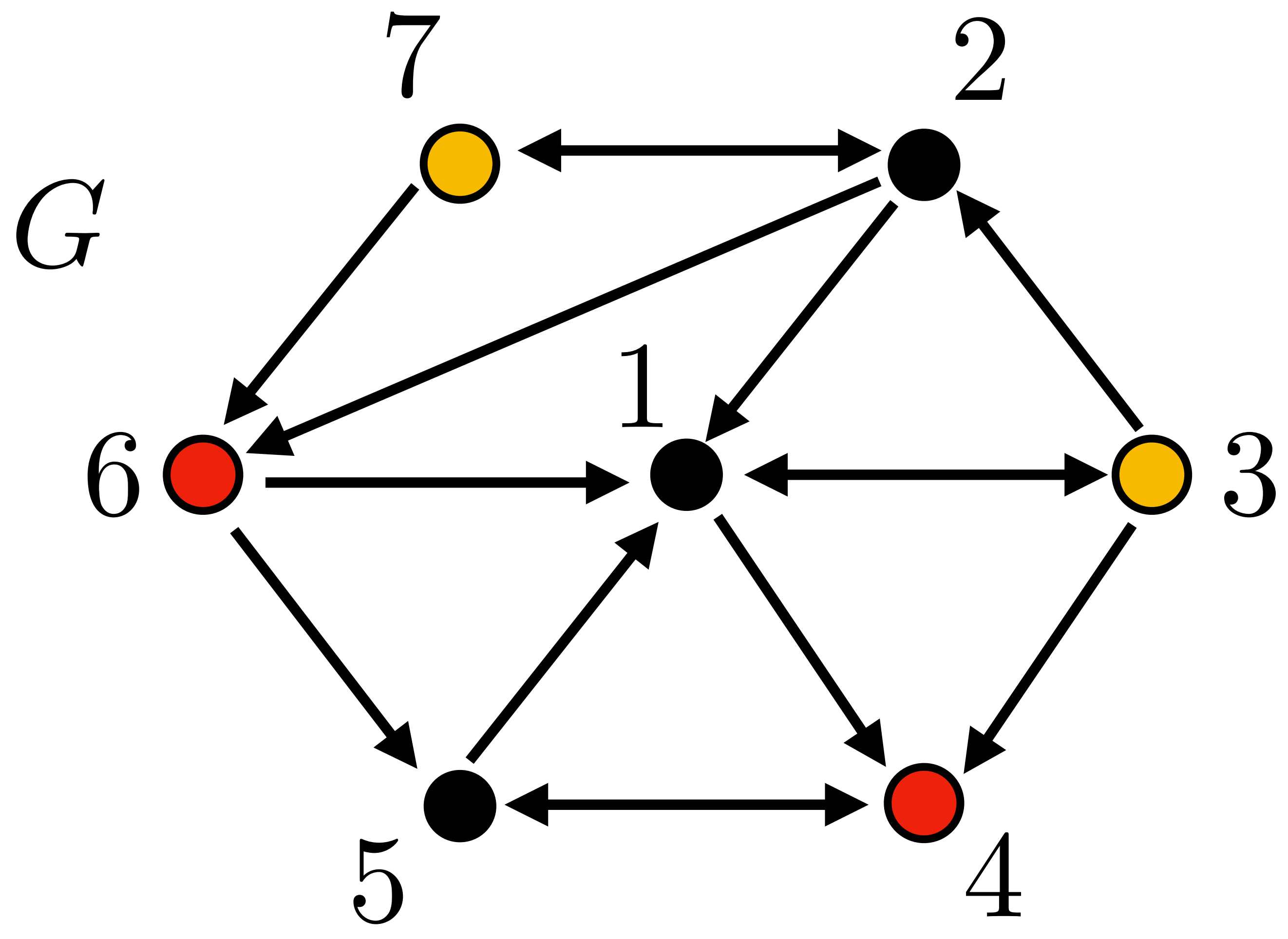
Example



$6 > 7$

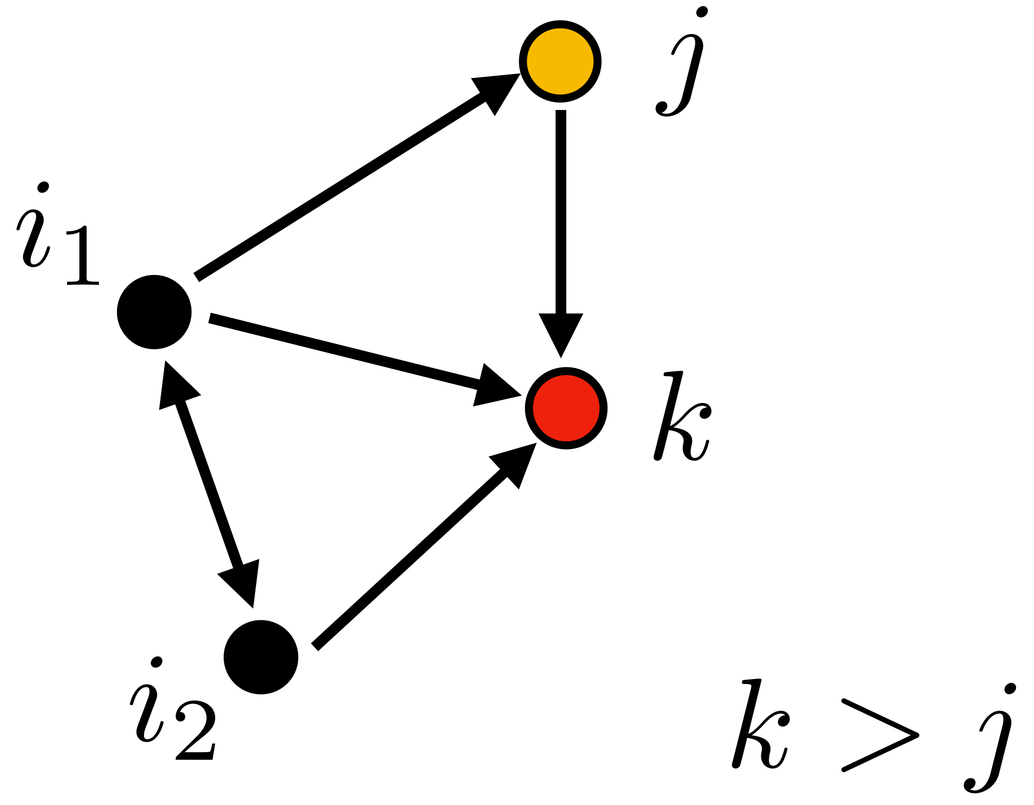
$4 > 3$

domination is a property of G

Domination

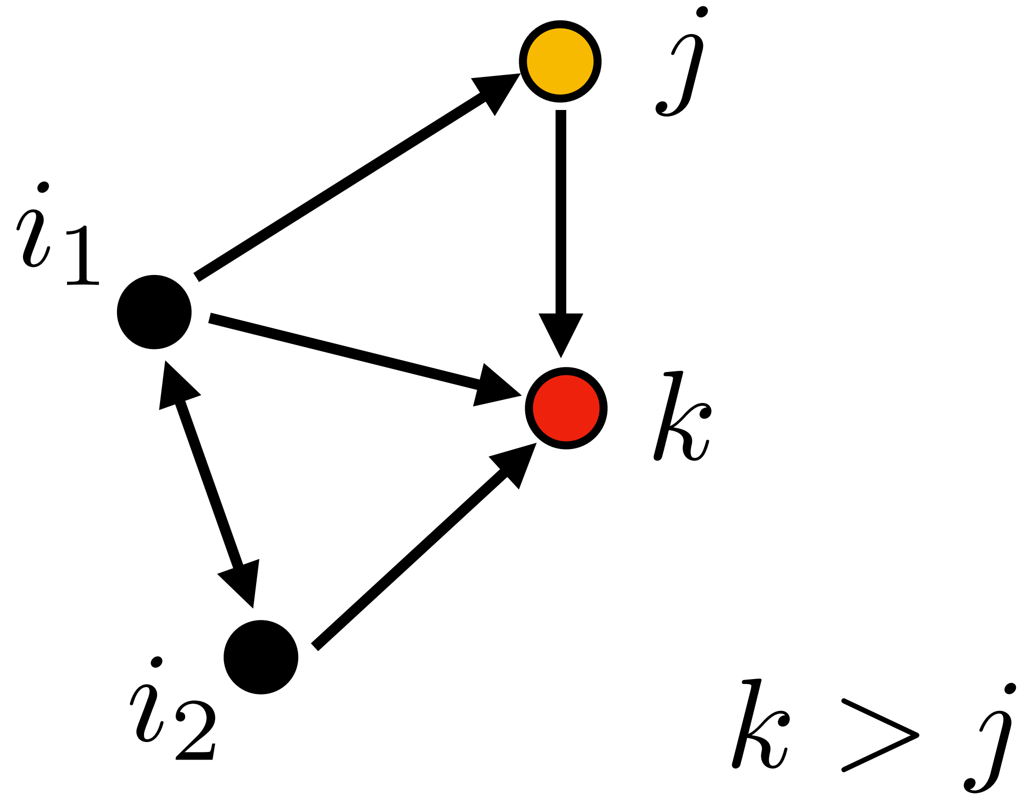
Old Theorem (2019)
If k dominates j in G, then j, k cannot both be active at any fixed point of a CTLN built from G.

$$\{j,k\} \not\subseteq \sigma \text{ for any } \sigma \in \mathrm{FP}(G)$$



Curto, Geneson, Morrison, 2019

Domination



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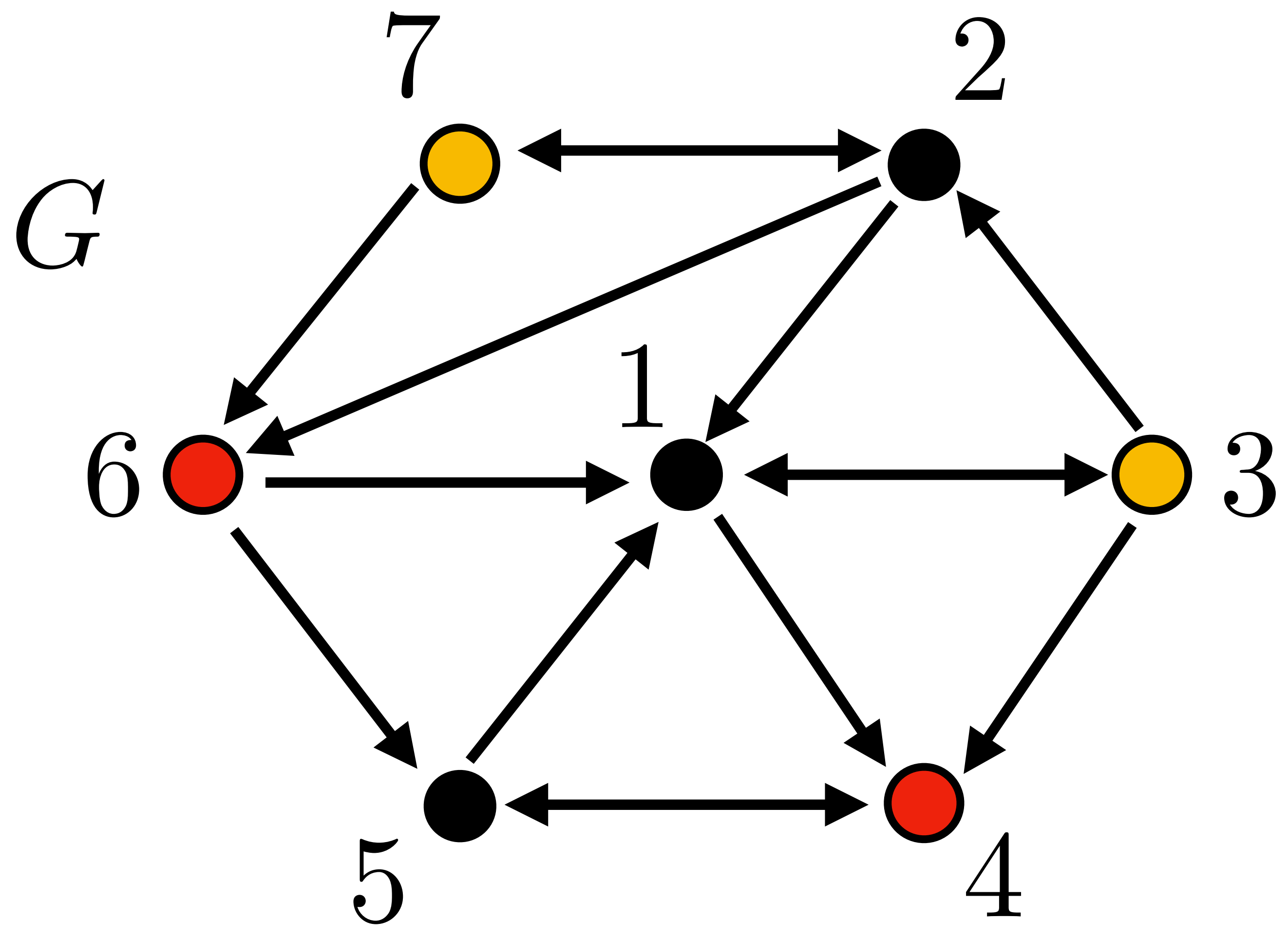


$6 > 7$

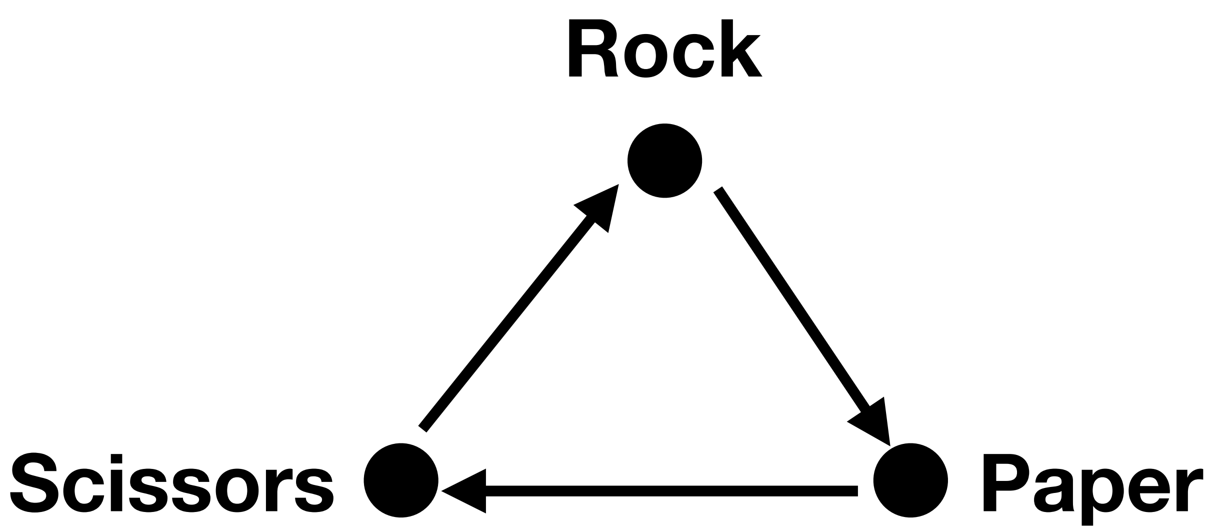
$4 > 3$

Old Theorem says: for any CTLN built from G, FP(G) cannot have any fixed points with both {6,7} or both {3,4}.

But it's not like we can remove 3 and 7; they may still affect or participate in other fixed points (for all we know).

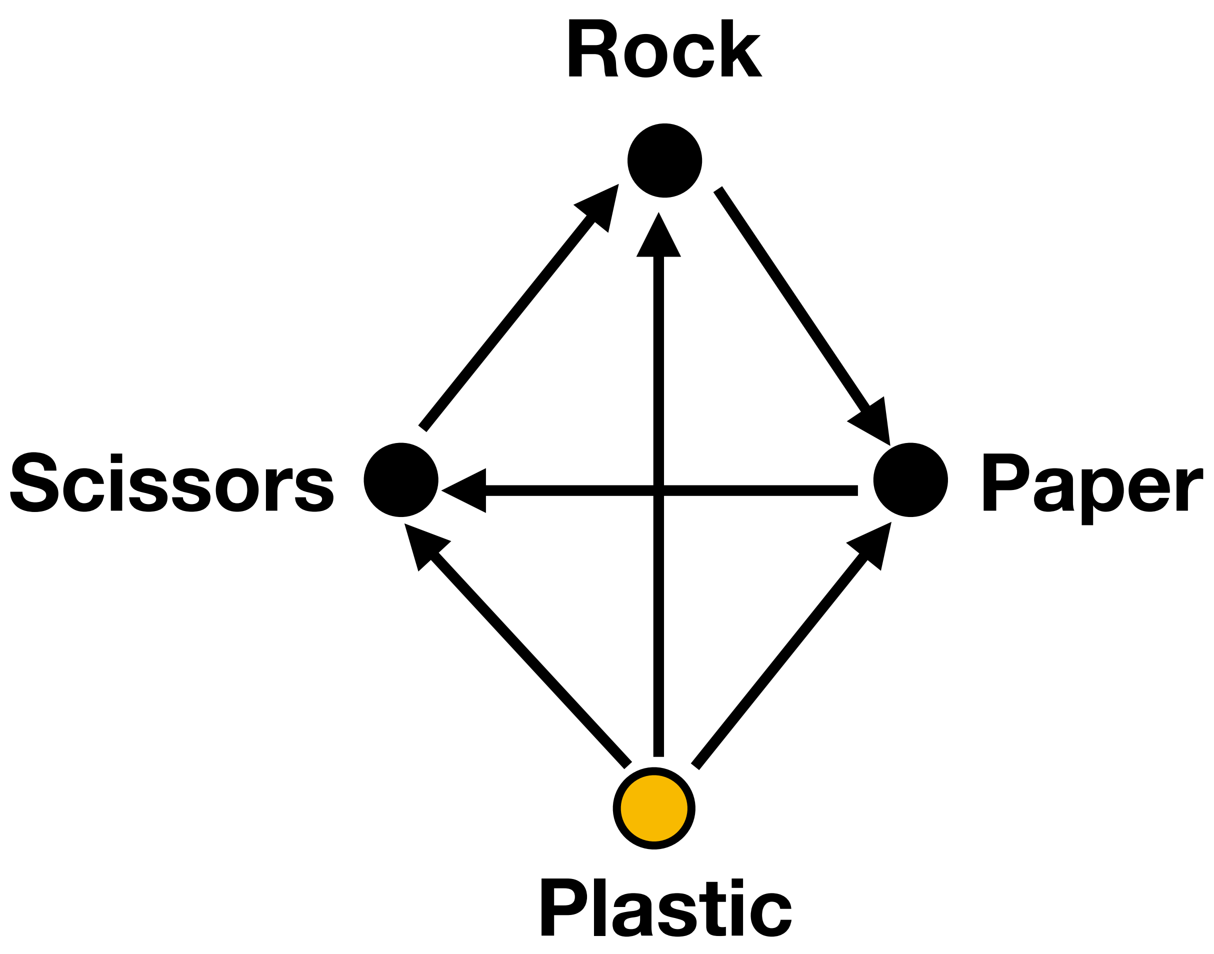
Curto, Geneson, Morrison, 2019

Rock-Paper-Scissors: a true story



March 2024

Rock-Paper-Scissors: a true story



Rock

Scissors

Paper

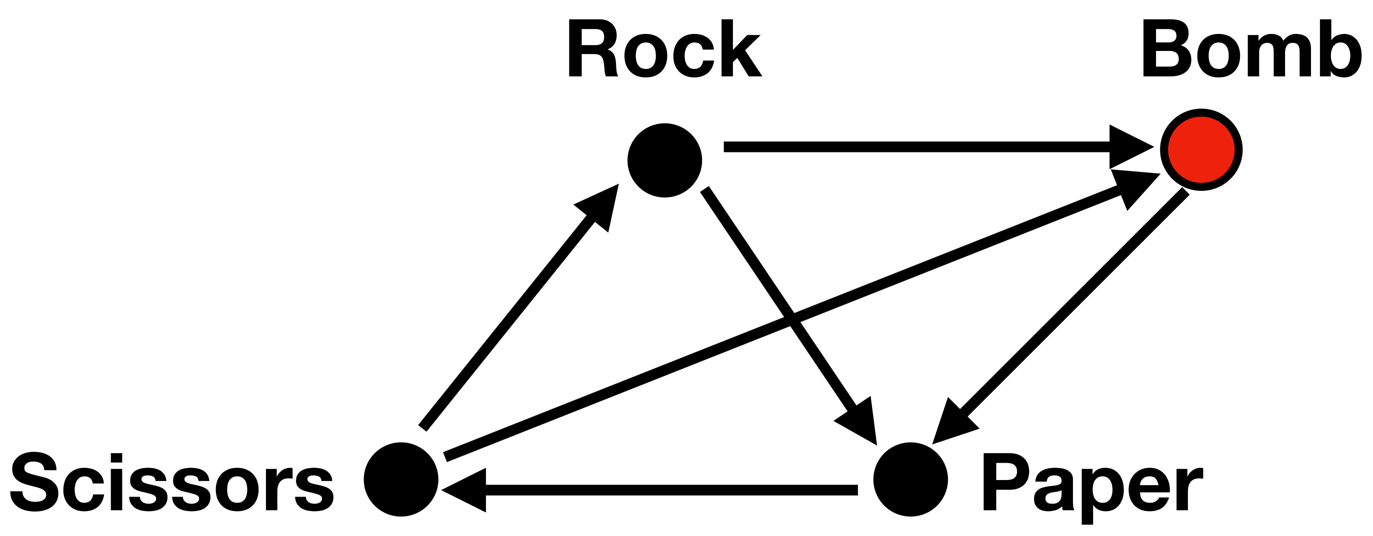
Plastic

Plastic loses to everyone, so nobody would ever pick it as a strategy.

It drops out.

March 2024

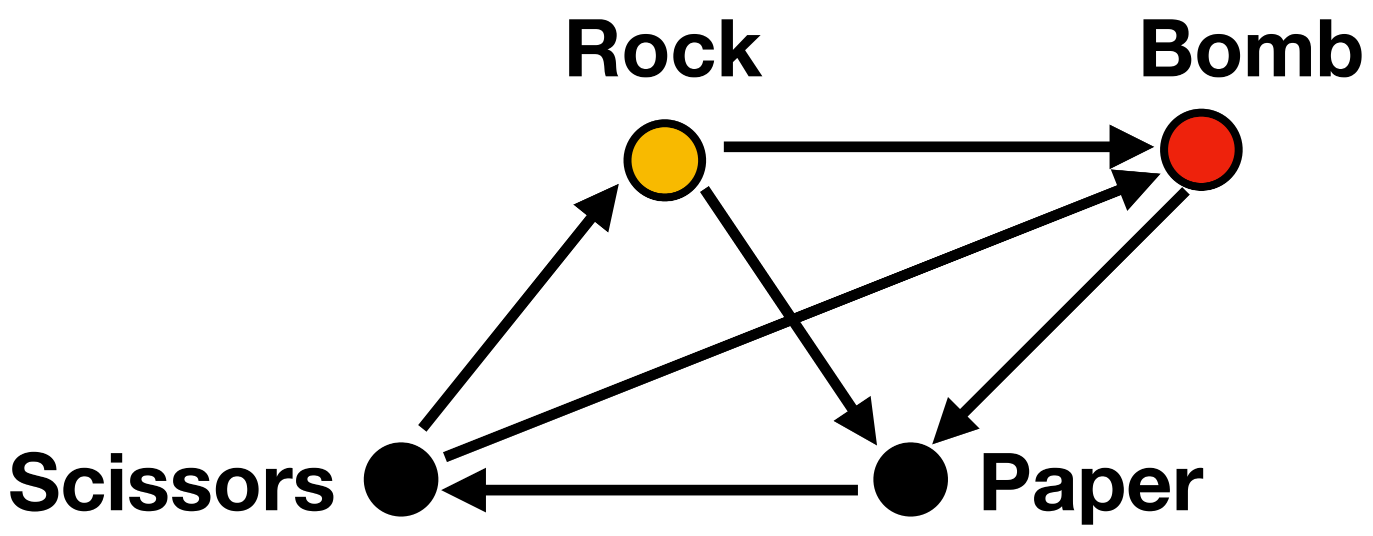
Rock-Paper-Scissors: a true story



Bomb beats Scissors and loses to Paper, just like Rock.
But Bomb also beats Rock.

March 2024

Rock-Paper-Scissors: a true story

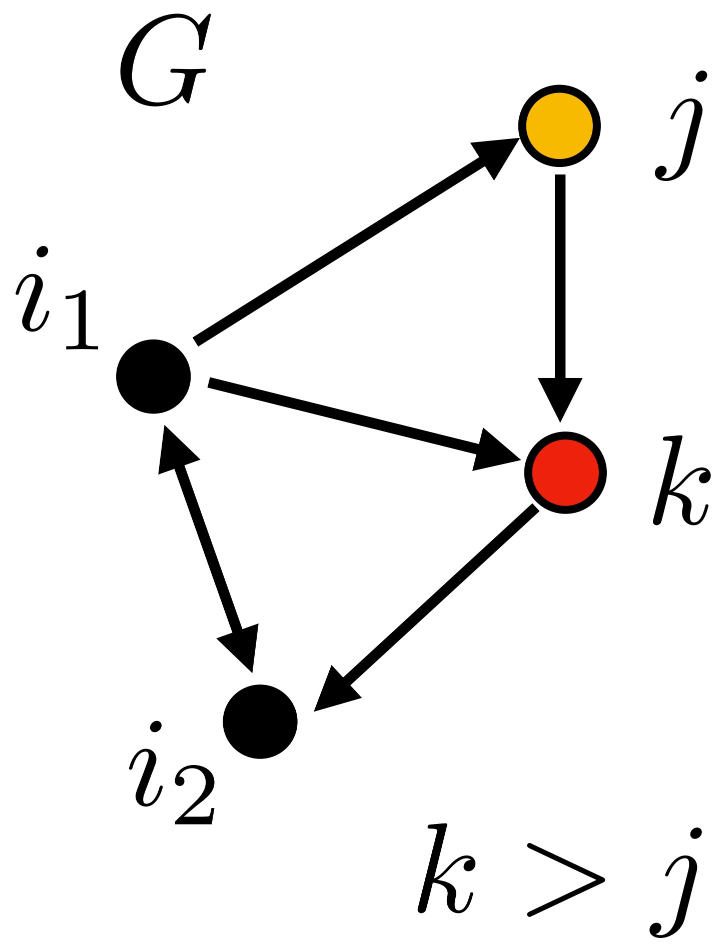


Bomb beats Scissors and loses to Paper, just like Rock.
But Bomb also beats Rock.

So now nobody would ever pick Rock as a strategy.
Rock drops out!

March 2024

Domination - New Theorems



Theorem 1 (2024)

If j is a dominated node in G, then it drops out!

I.e., in any gCTLN, we have: $\mathrm{FP}(G) = \mathrm{FP}(G|_{[n]\setminus j})$

Curto 2024 (unpublished)

Domination – New Theorems

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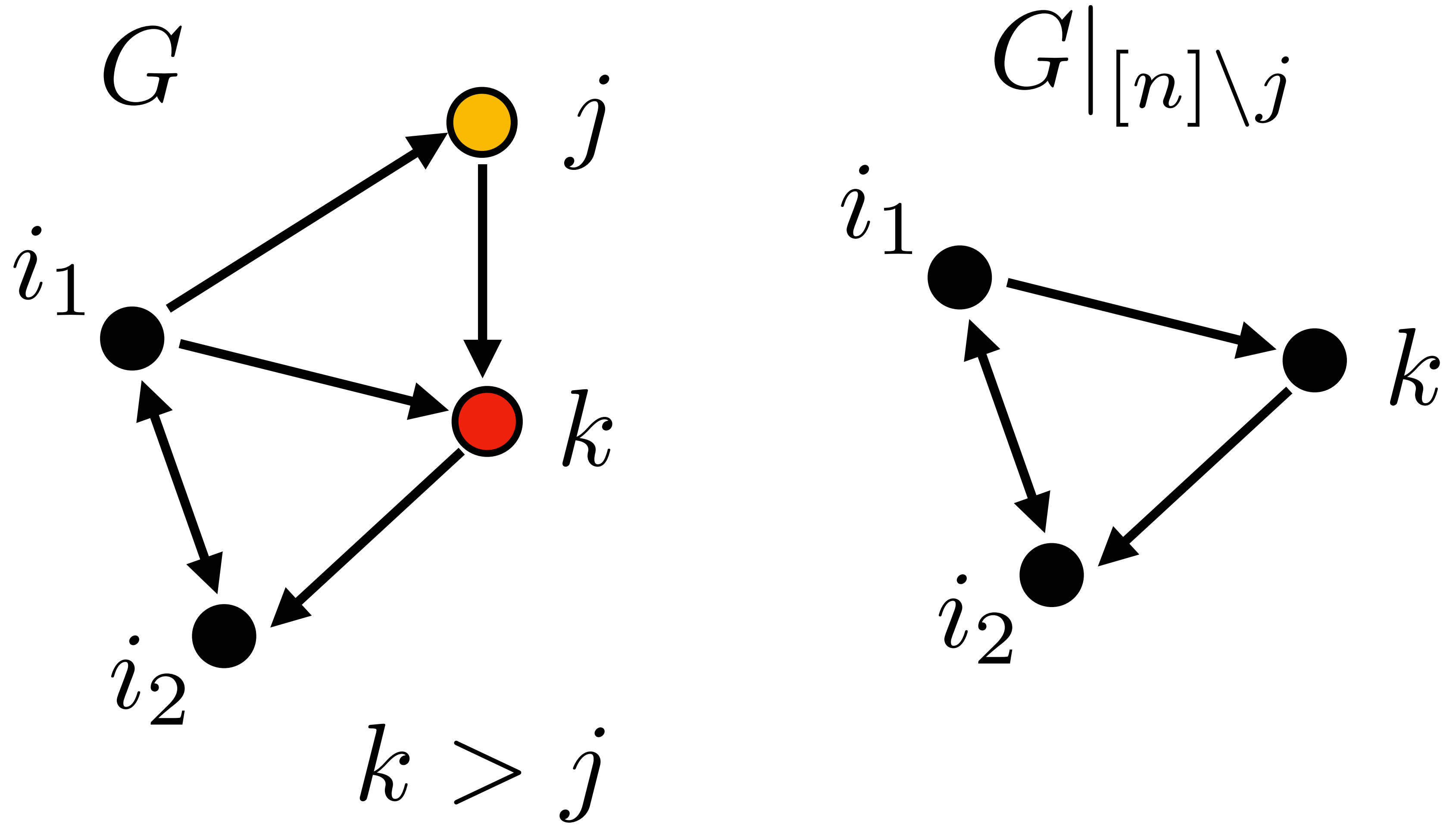
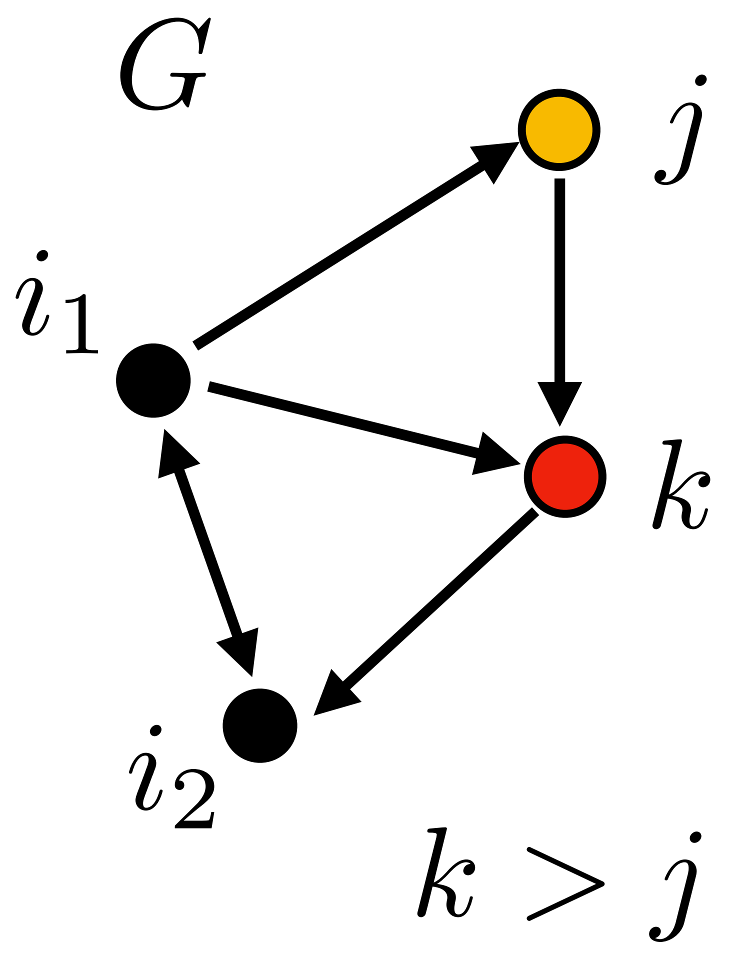
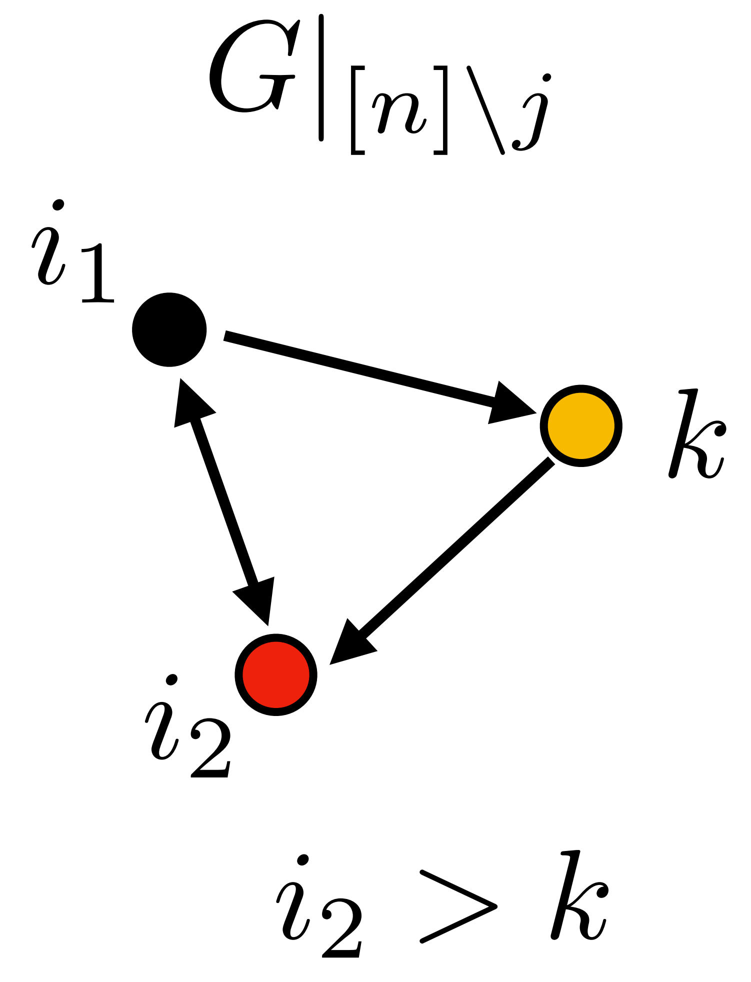
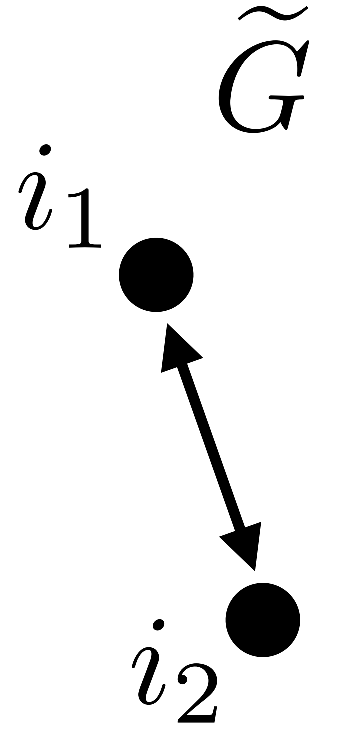


Curto 2024 (unpublished)

Domination – New Theorems







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I.e., in any gCTLN, we have: $\text{FP}(G) = \text{FP}(G|_{[n]\setminus j})$

Theorem 2 (2024)

By iteratively removing dominated nodes, the final reduced graph G-tilde is unique. Moreover, $\text{FP}(G) = \text{FP}(\widetilde{G})$

Curto 2024 (unpublished)

Domination – New Theorems

Theorem 1 (2024)

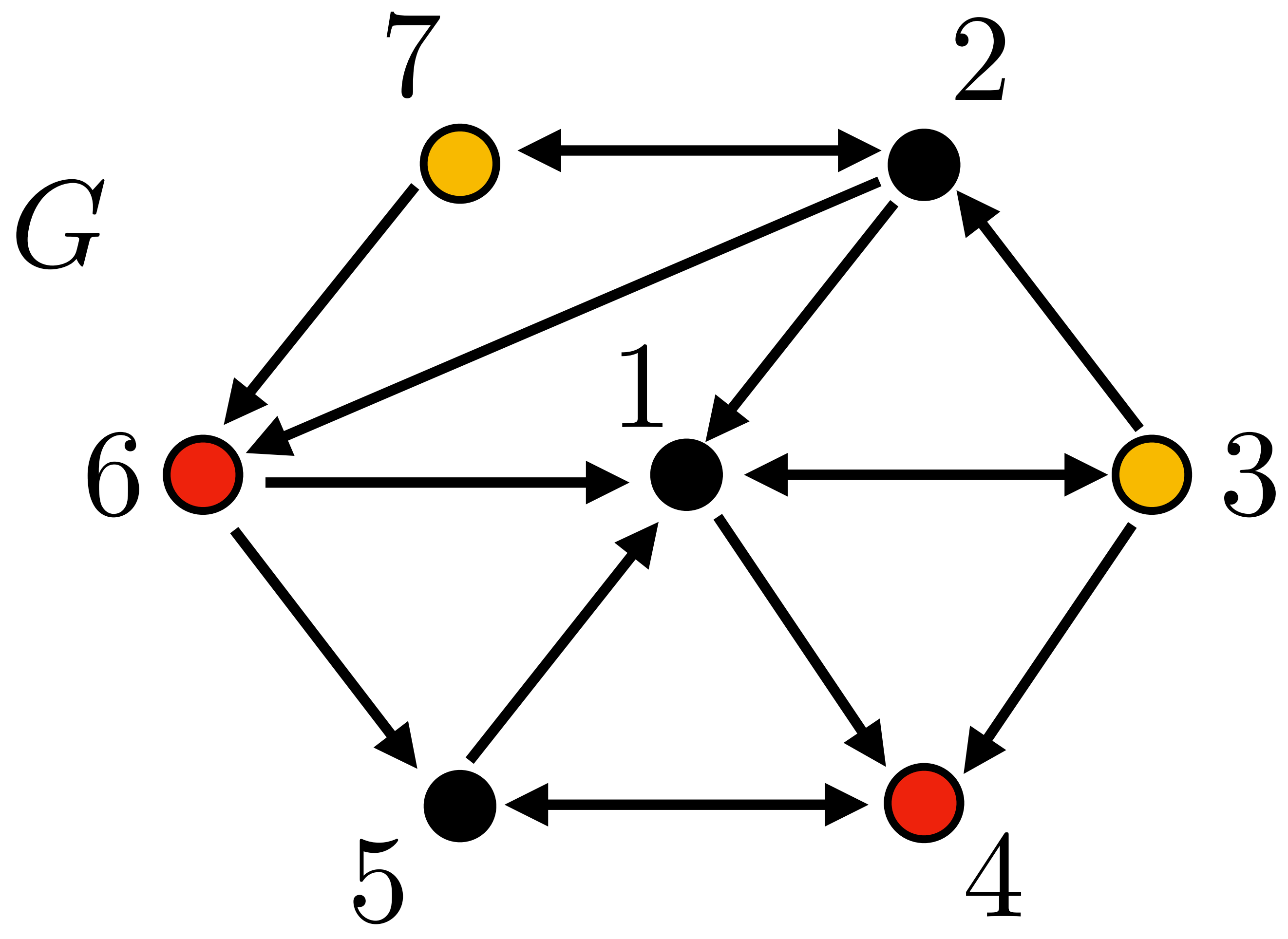
If j is a dominated node in G, then it drops out!

I.e., in any gCTLN, we have: $$\mathrm{FP}(G) = \mathrm{FP}(G|_{[n]\setminus j})$$

Theorem 2 (2024)

By iteratively removing dominated nodes, the final reduced graph G-tilde is unique. Moreover, $$\mathrm{FP}(G) = \mathrm{FP}(\widetilde{G})$$

Example



$6 > 7$ $\quad$ $4 > 3$

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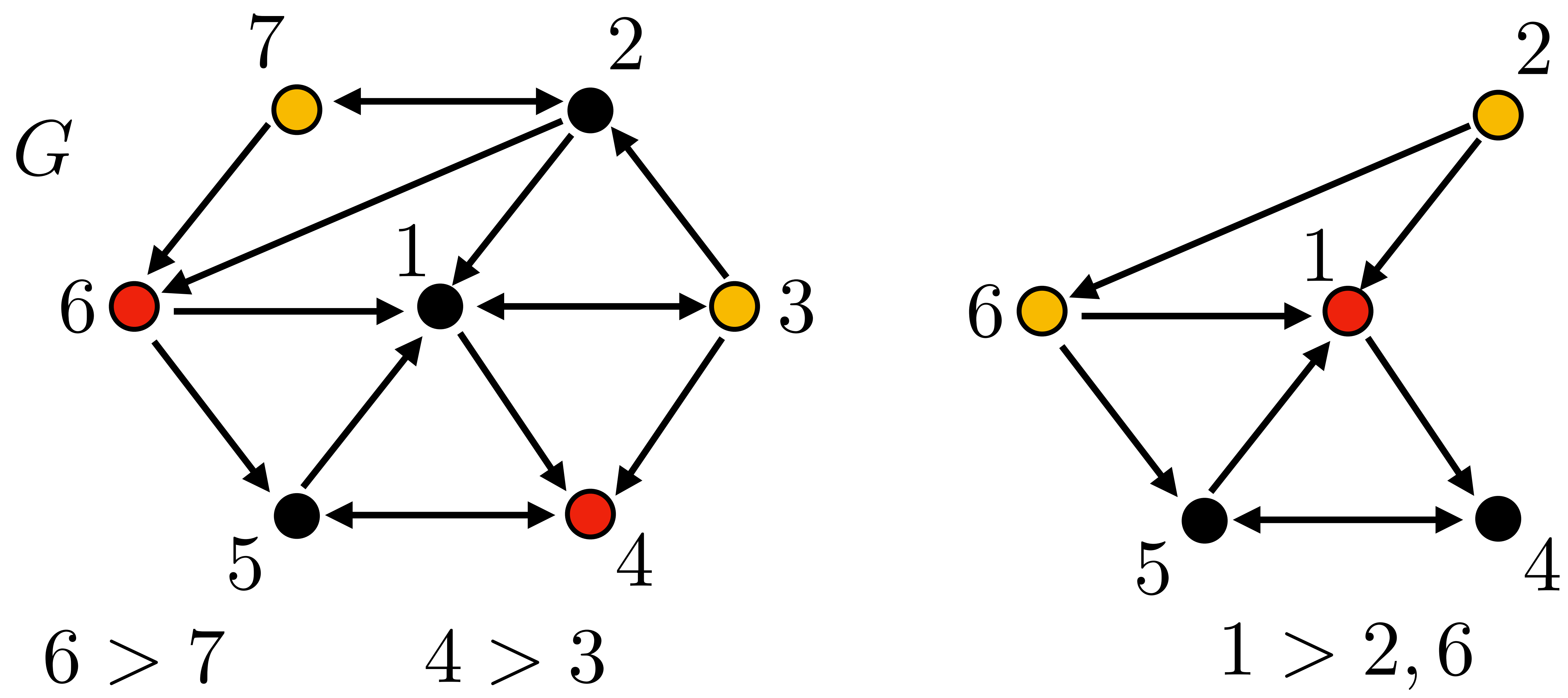

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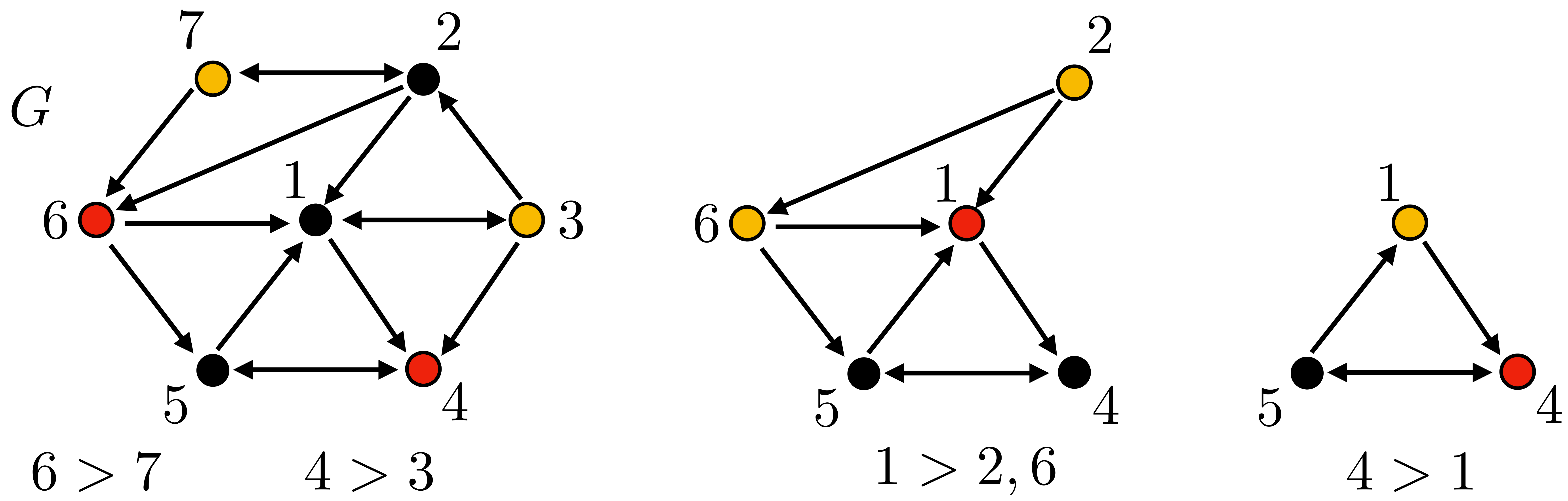

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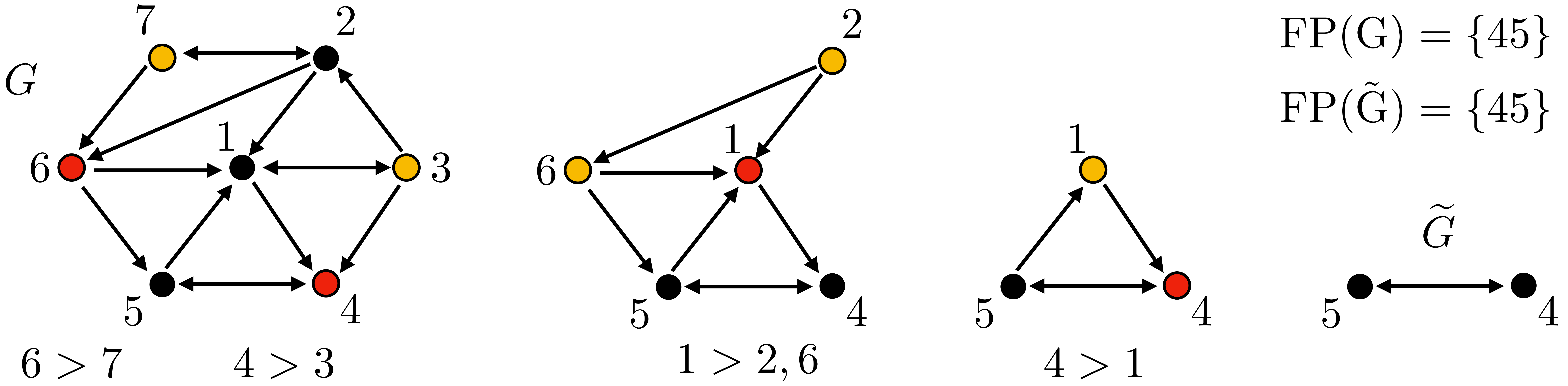

$\mathrm{FP(G)} = \{45\}$

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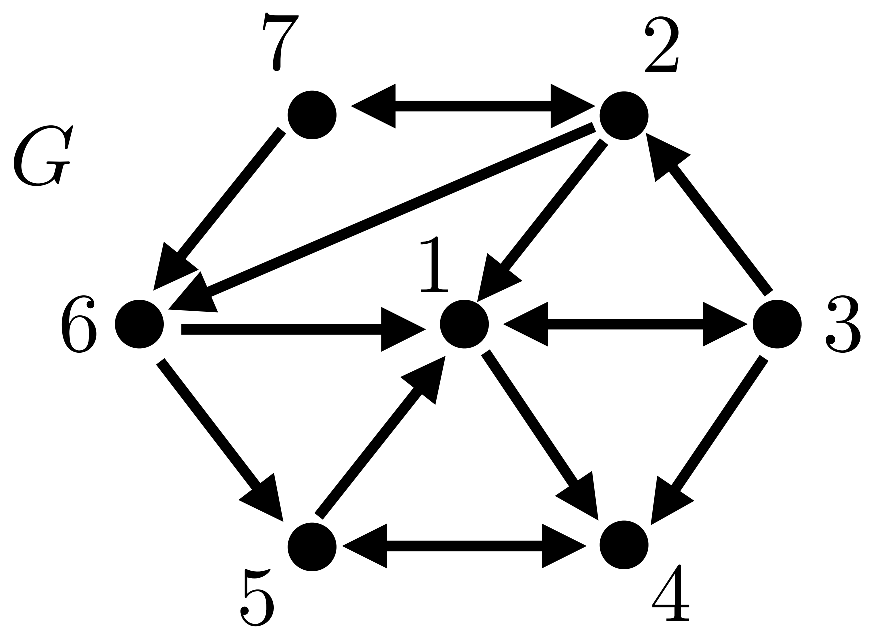
Computational Experiments

Example



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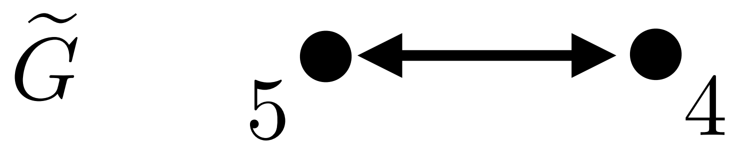

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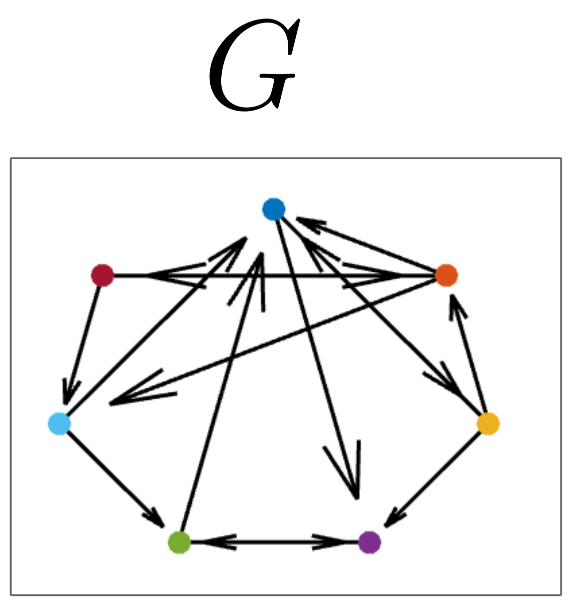

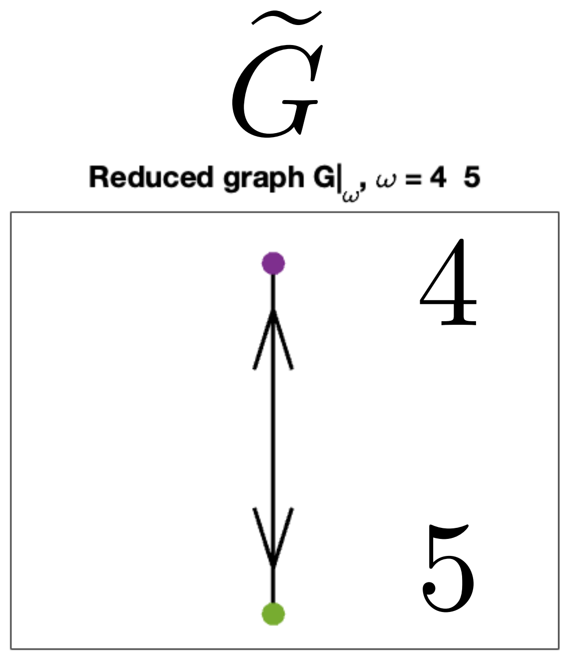

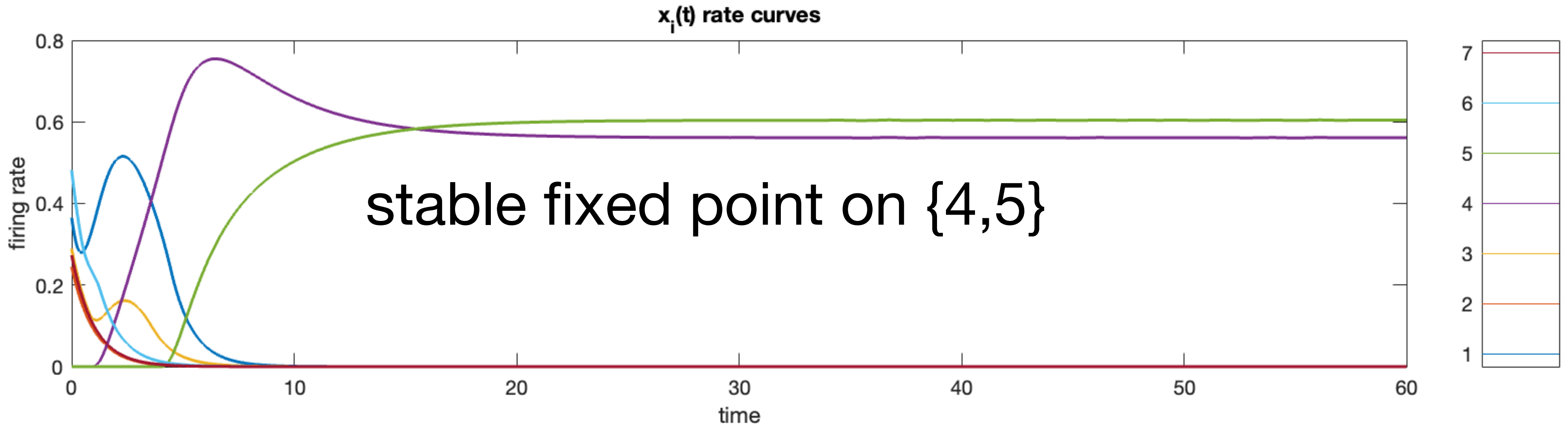

stable fixed point on {4,5}

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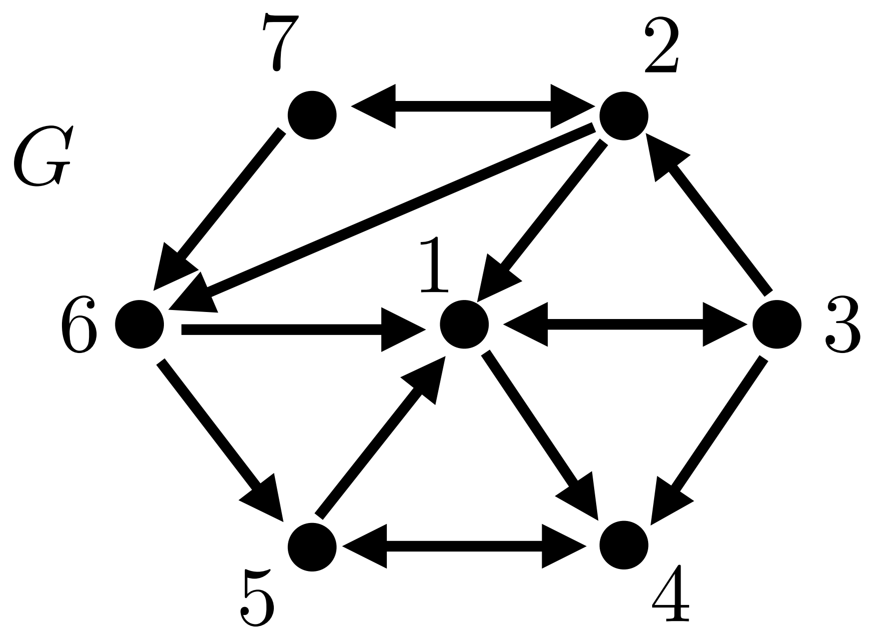

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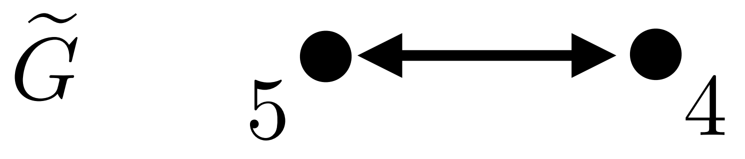

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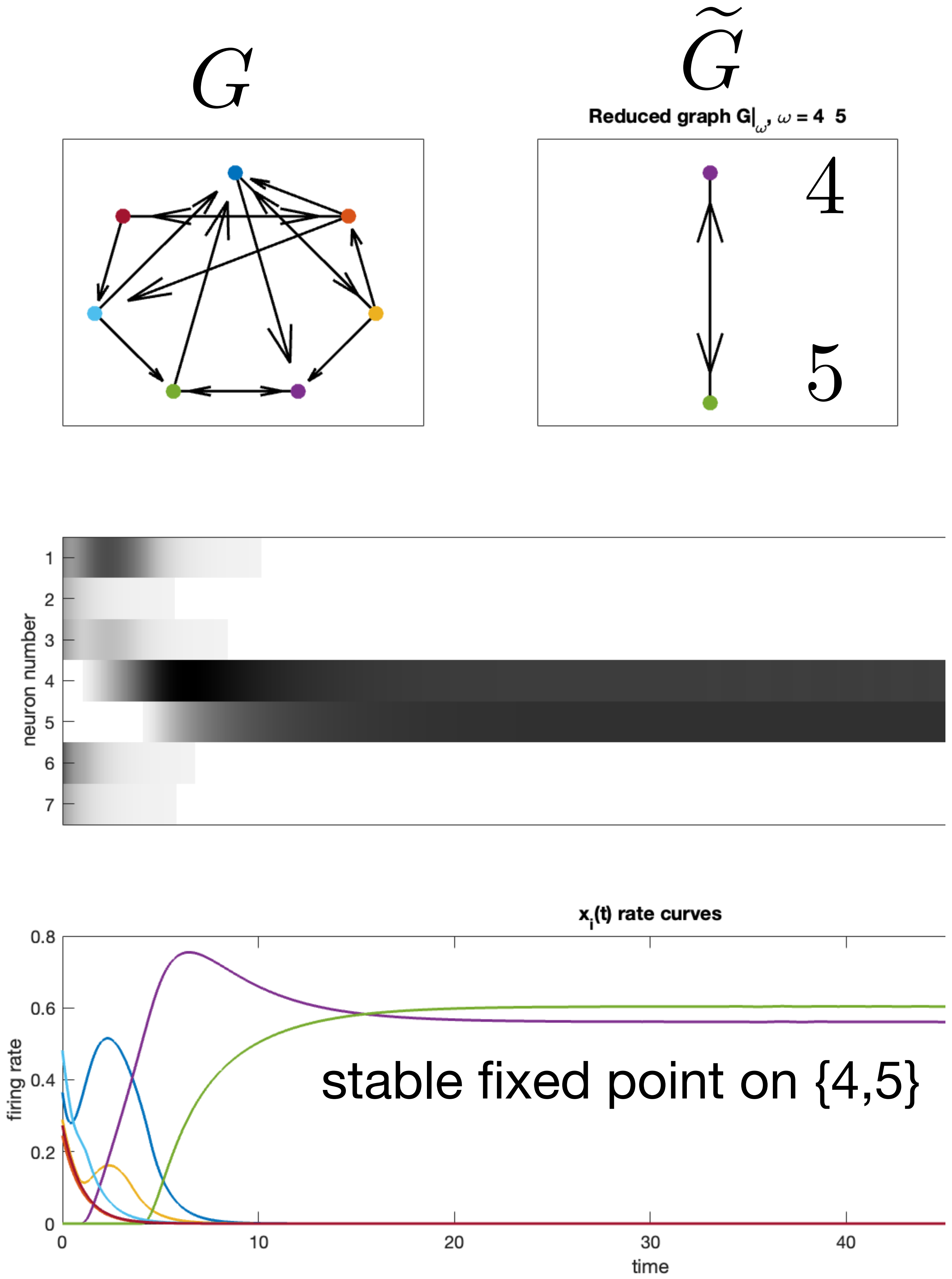
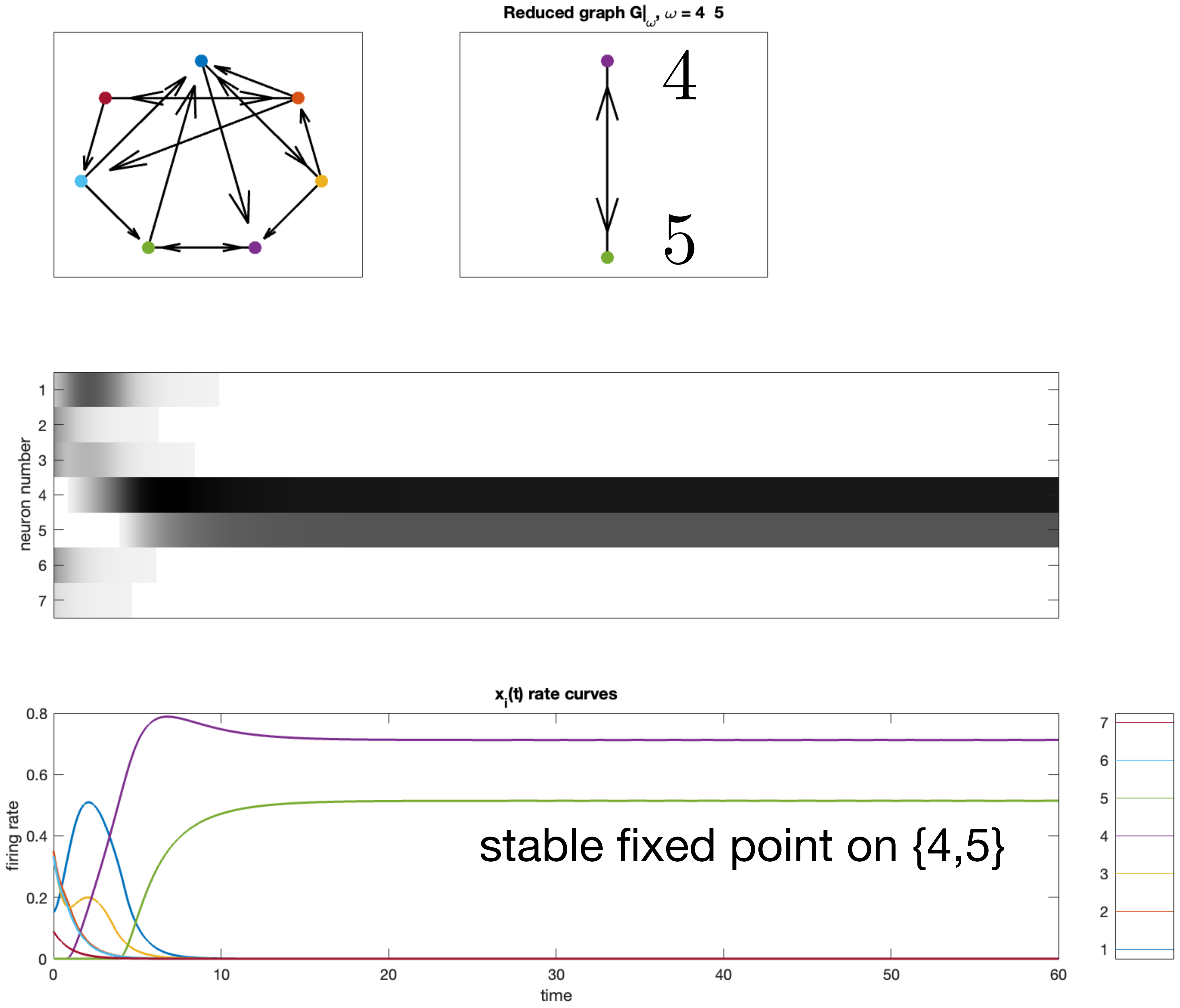


stable fixed point on {4,5}

same graph, different gCTLN parameters



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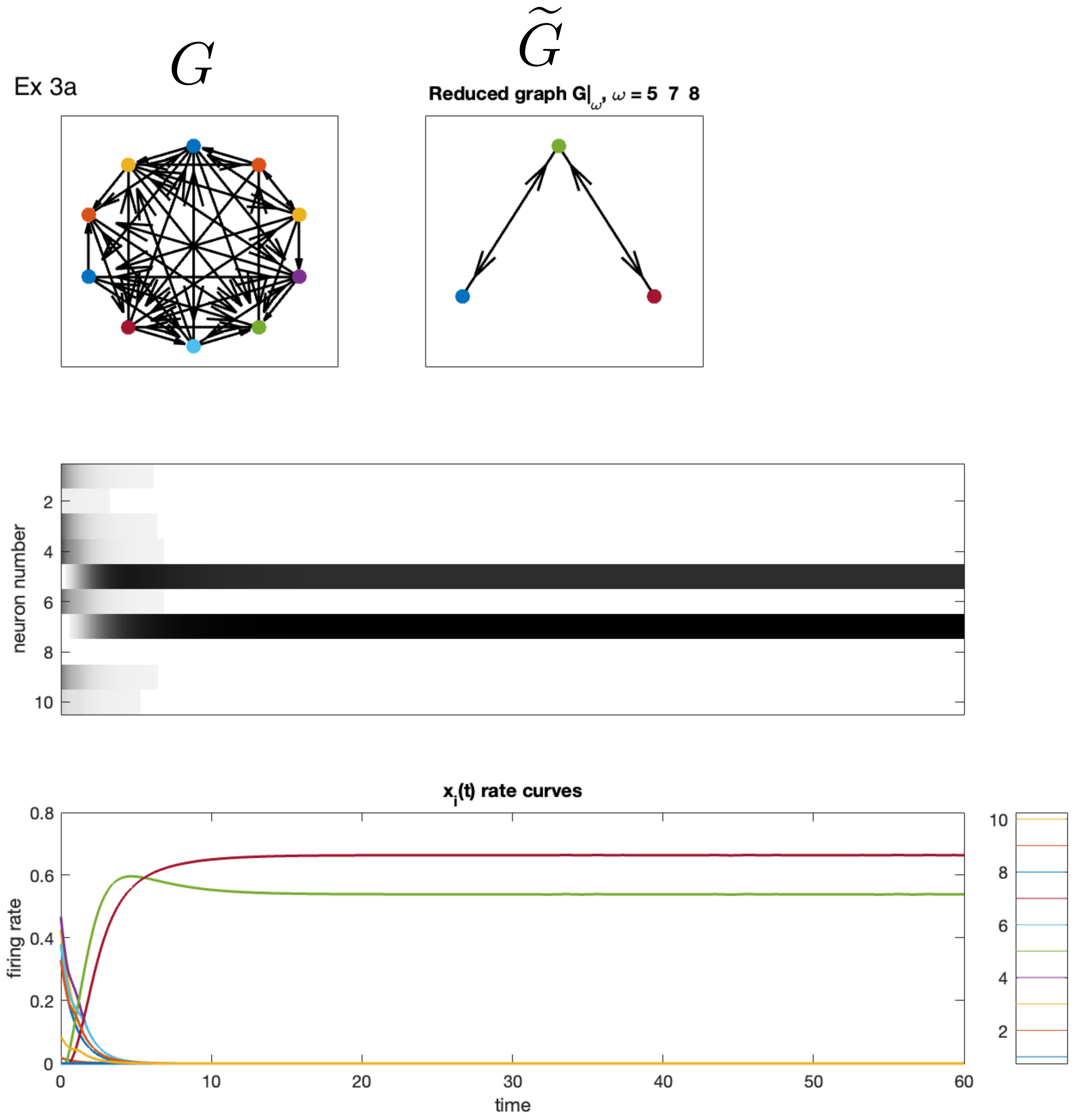
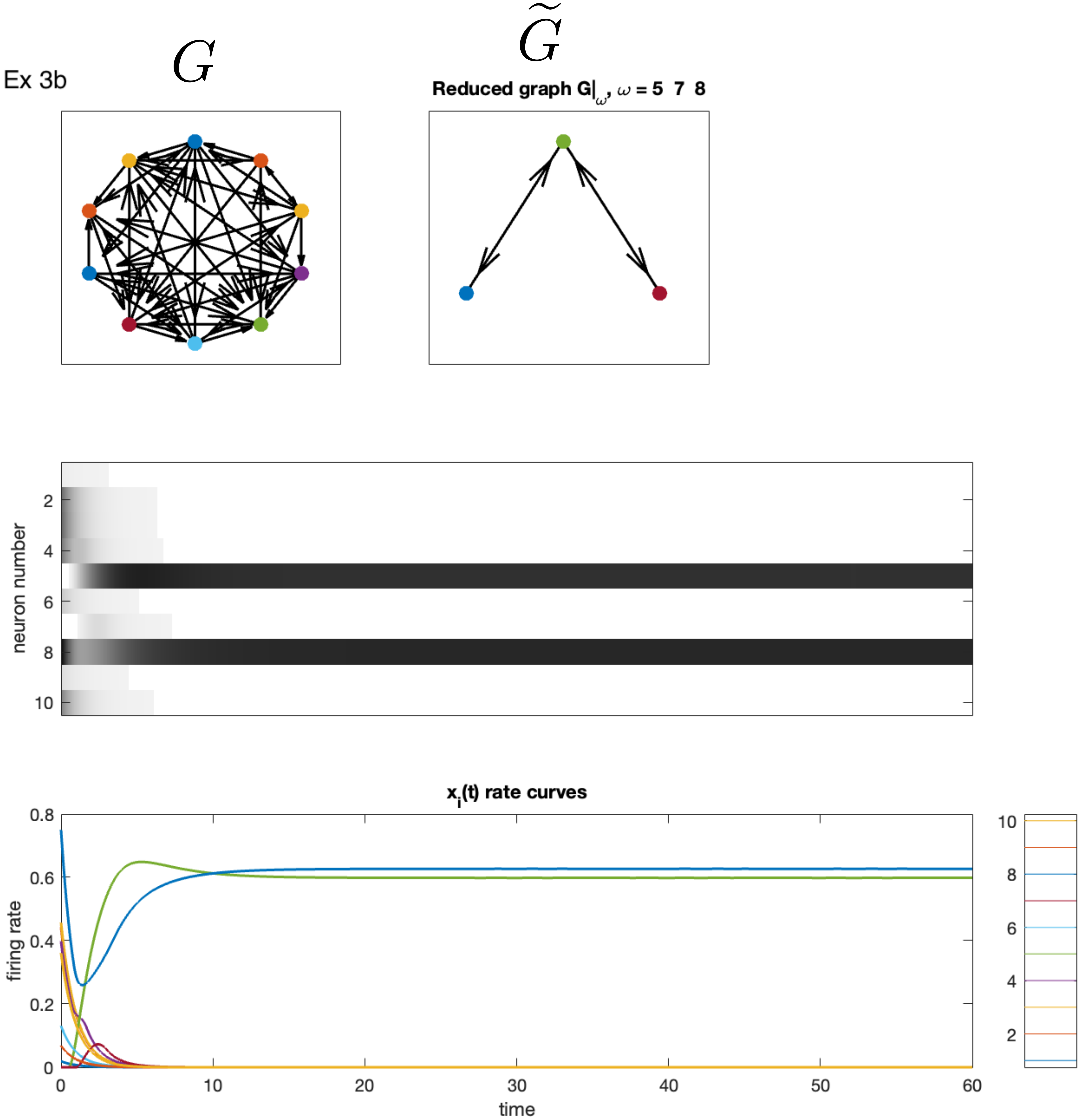
<u>Conjecture</u>: network activity flows from any initial condition on the graph to the reduced network $\widetilde{G}$

E-R random graphs with p=0.5





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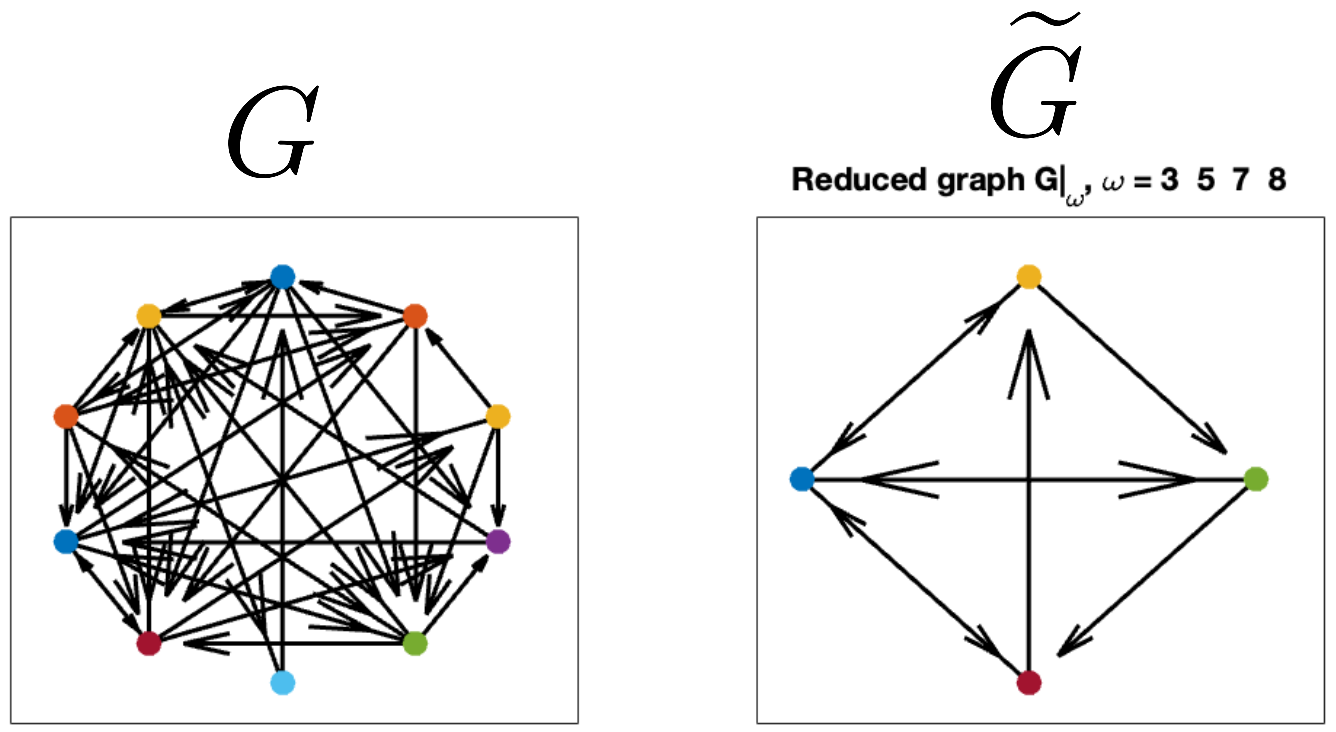
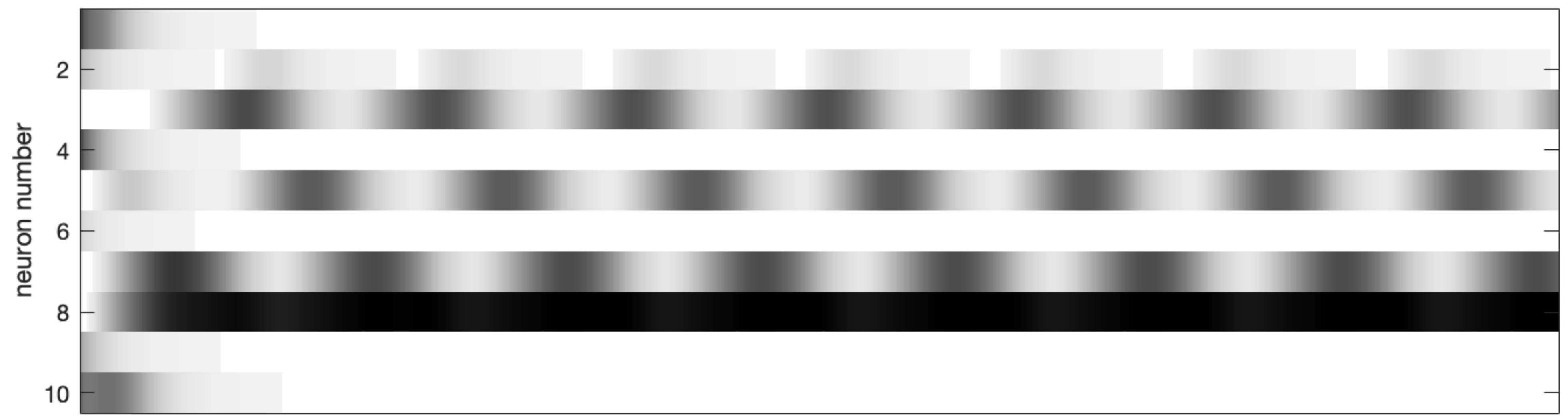
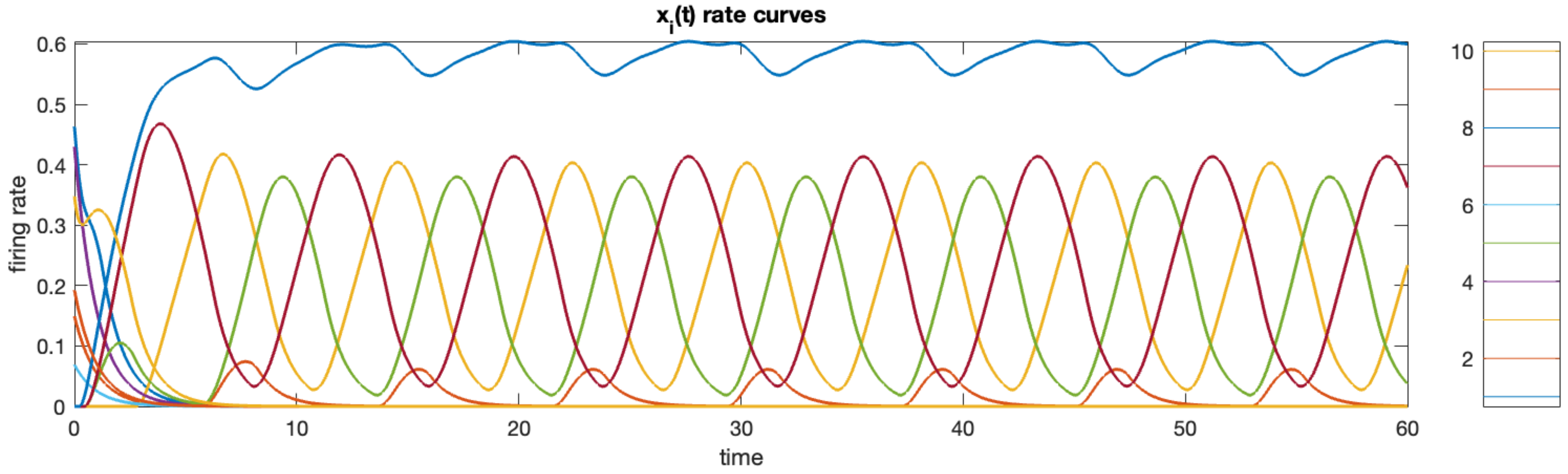
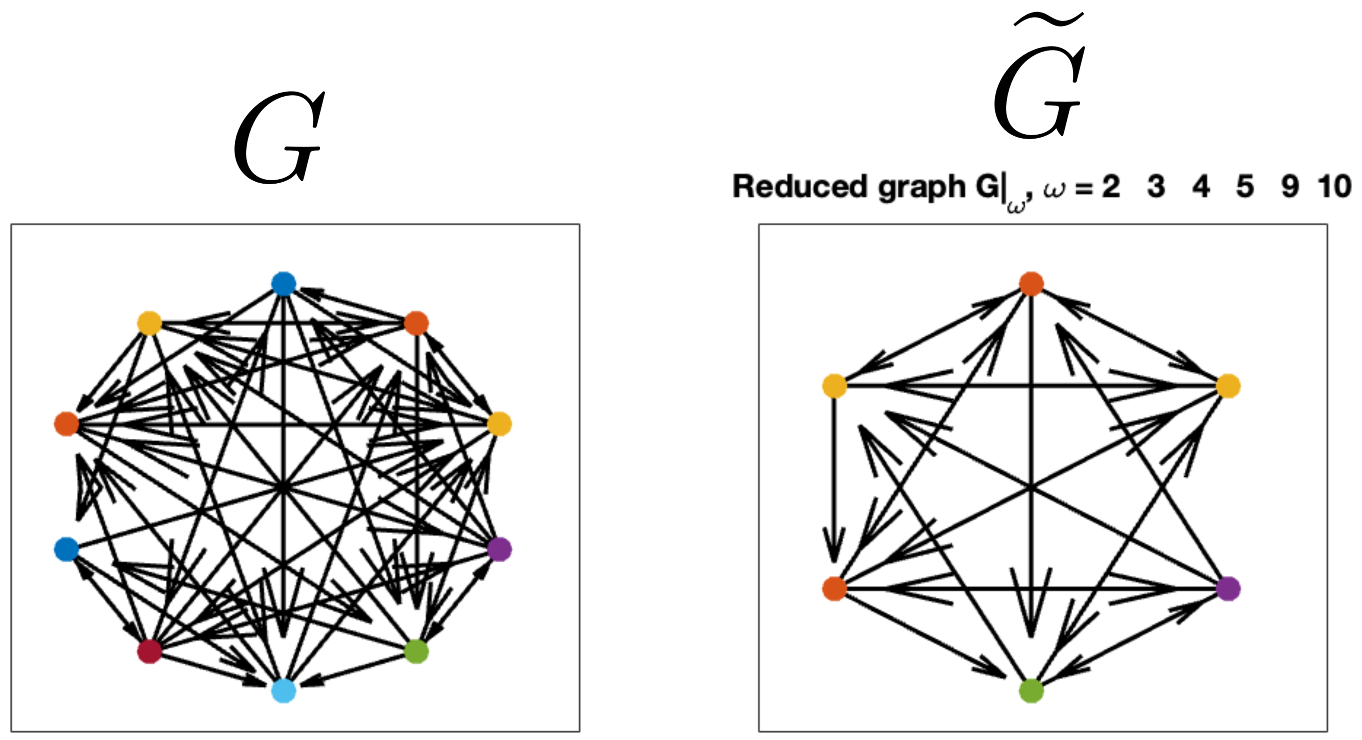
Ex 6

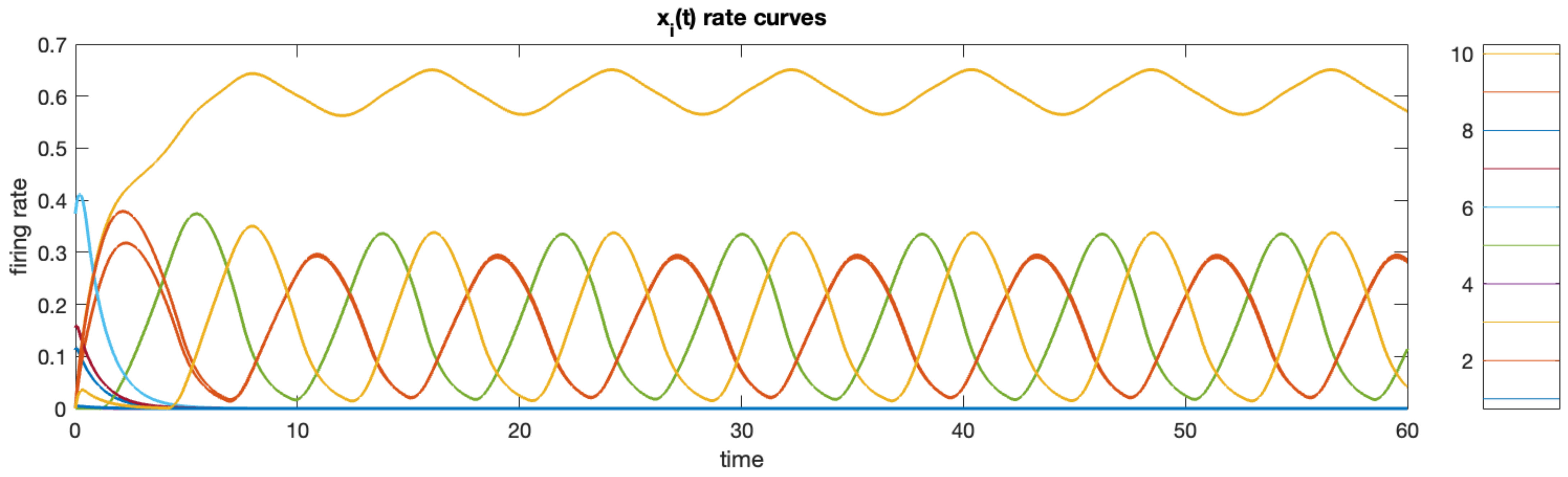




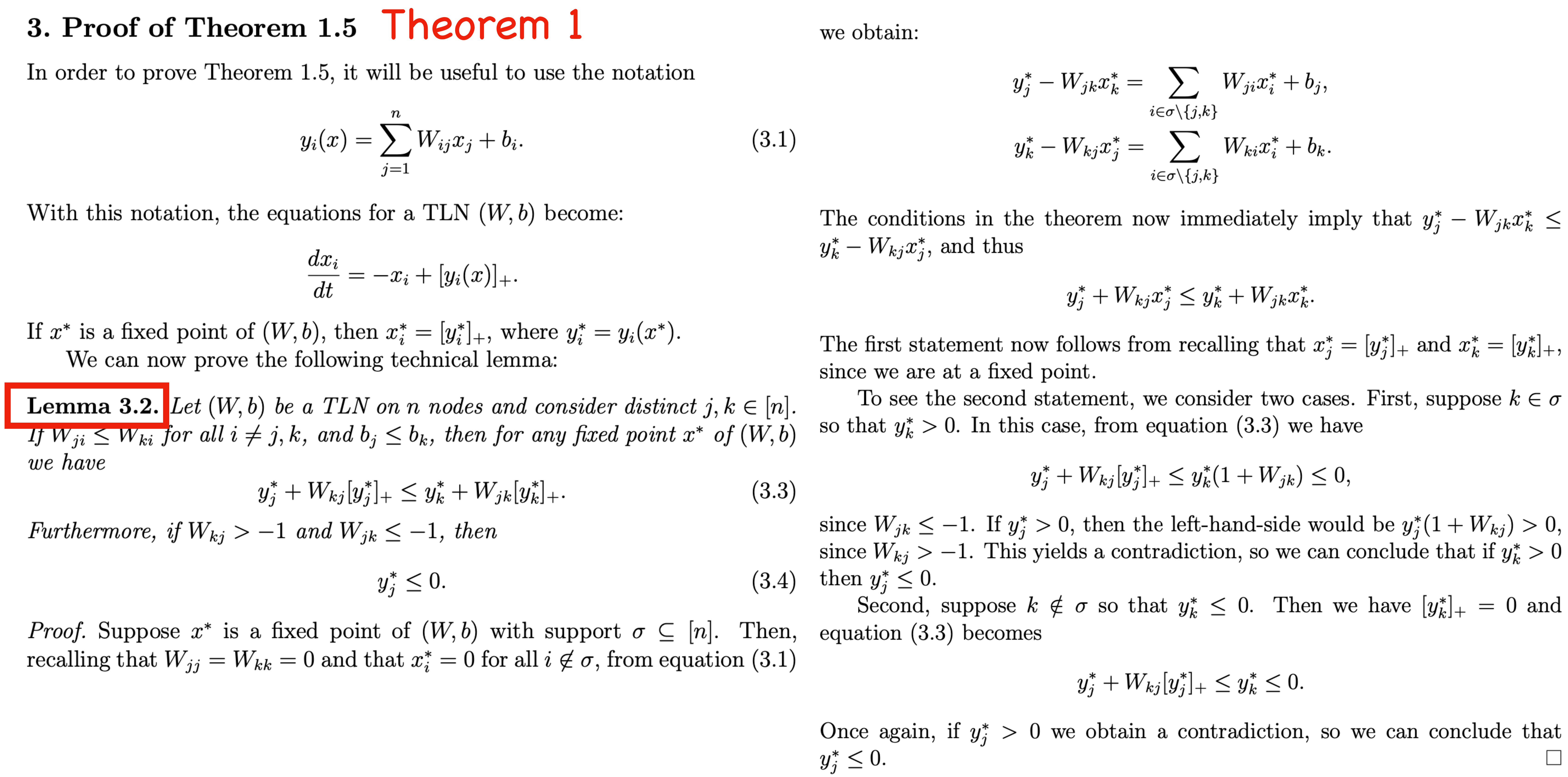


Ex 9





Domination - New Theorems - a word about the proofs



3. Proof of Theorem 1.5

Theorem 1

In order to prove Theorem 1.5, it will be useful to use the notation

$$y_i(x) = \sum_{j=1}^{n} W_{ij} x_j + b_i. \tag{3.1}$$

With this notation, the equations for a TLN (W, b) become:

$$\frac{dx_i}{dt} = -x_i + [y_i(x)]_+.$$

If x^* is a fixed point of (W, b), then $x_i^* = [y_i^*]_+$, where $y_i^* = y_i(x^*)$.

We can now prove the following technical lemma:

Lemma 3.2. *Let (W, b) be a TLN on n nodes and consider distinct $j, k \in [n]$. If $W_{ji} \leq W_{ki}$ for all $i \neq j, k$, and $b_j \leq b_k$, then for any fixed point x^* of (W, b) we have*

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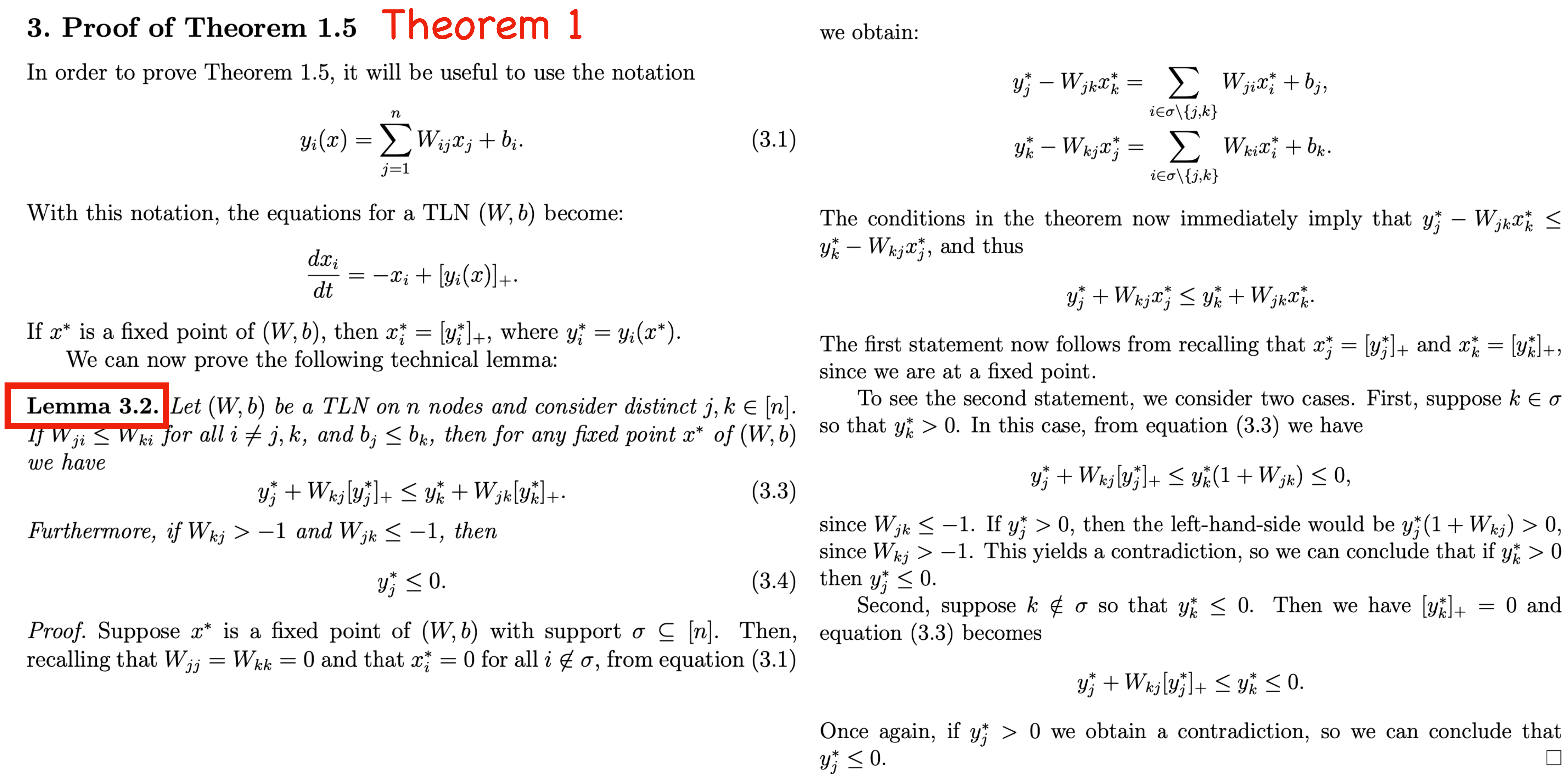

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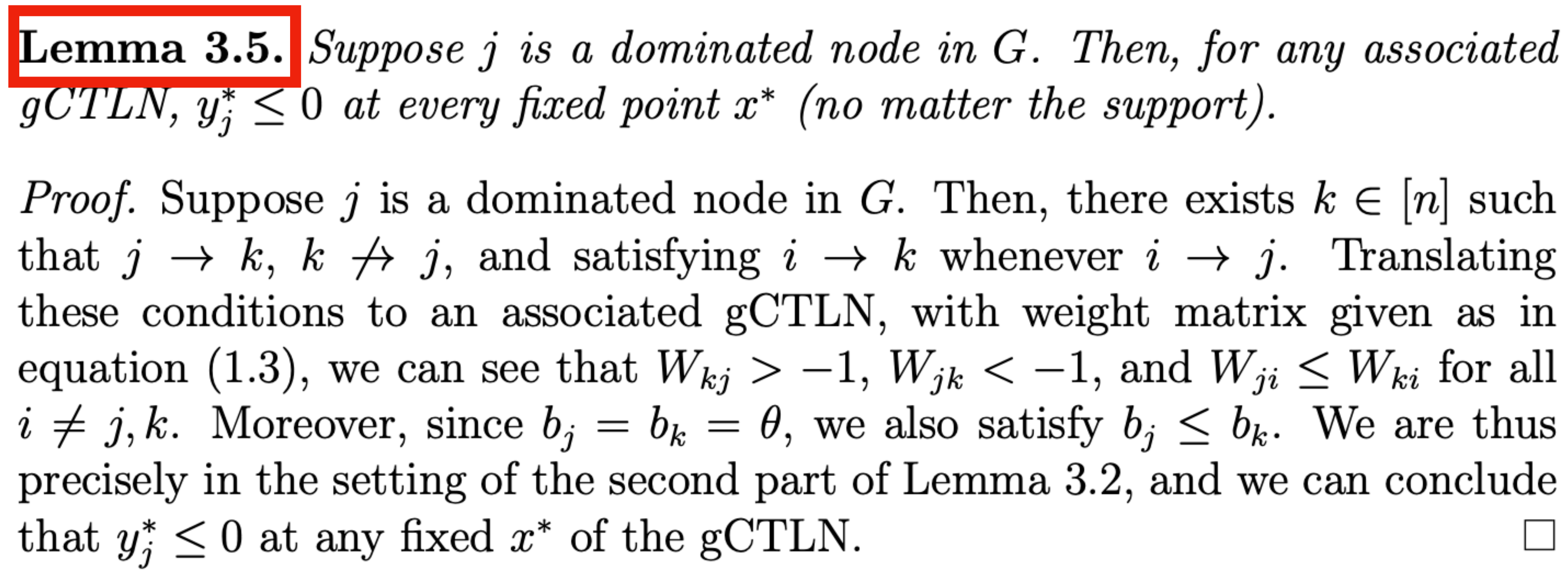

Lemma 3.5. *Suppose j is a dominated node in G. Then, for any associated gCTLN, $y_j^* \leq 0$ at every fixed point x^* (no matter the support).*

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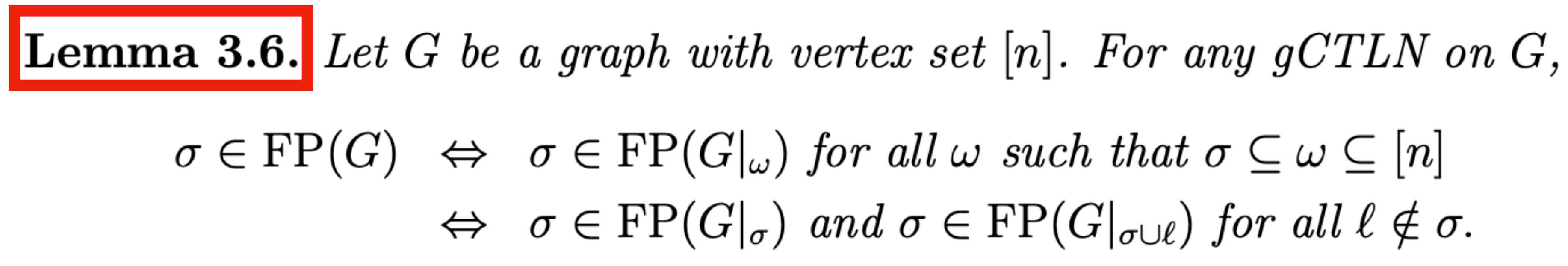
need some more lemmas…



Lemma 3.6. *Let G be a graph with vertex set $[n]$. For any gCTLN on G,*

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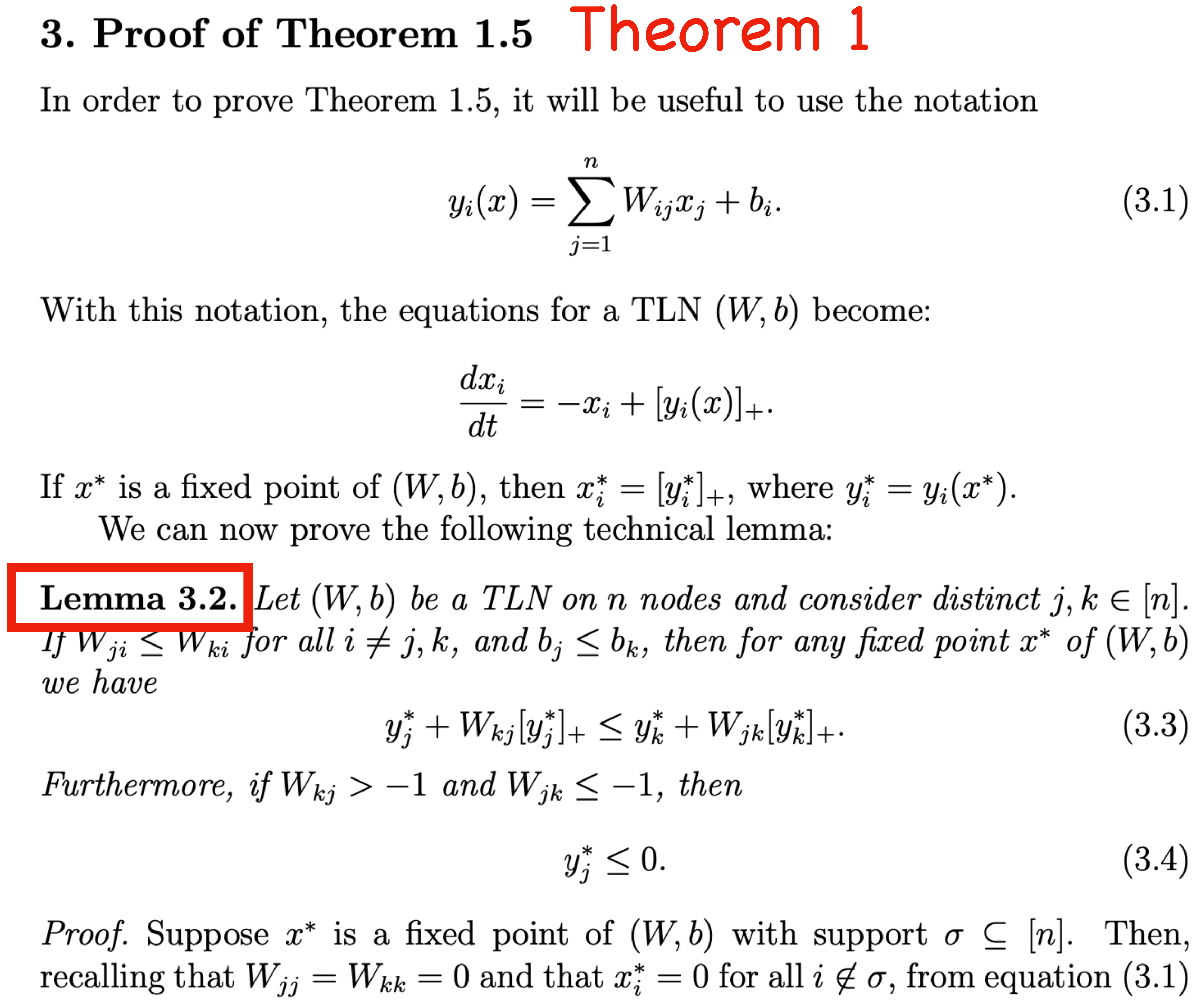

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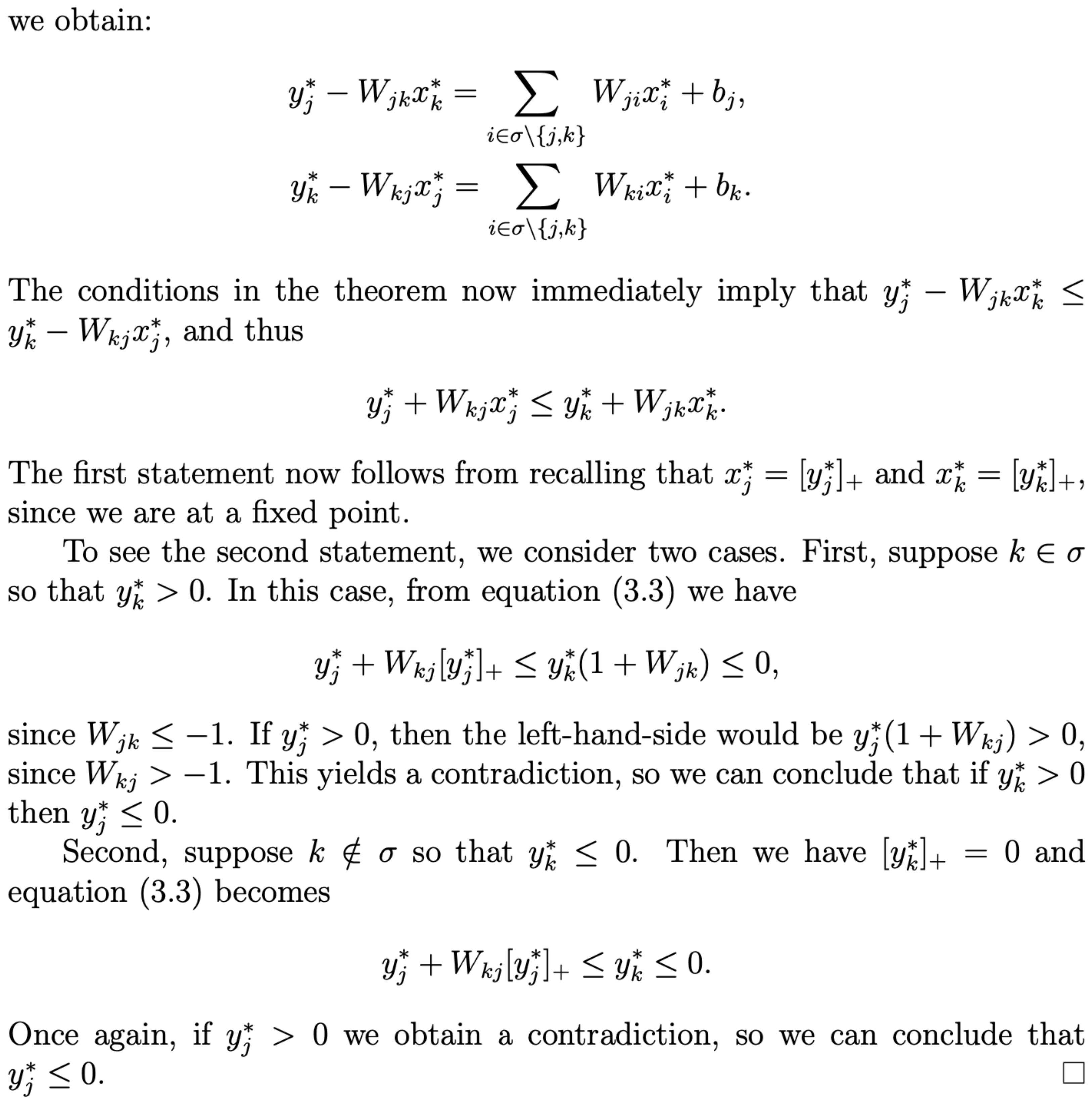

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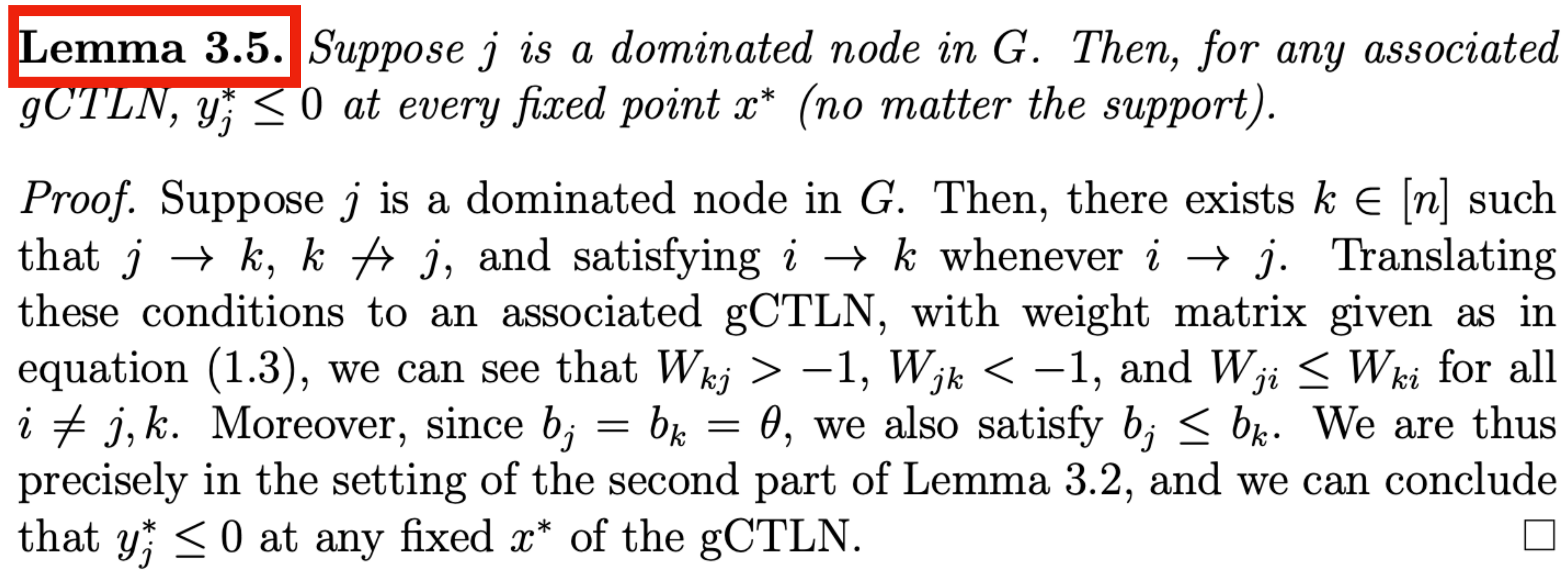

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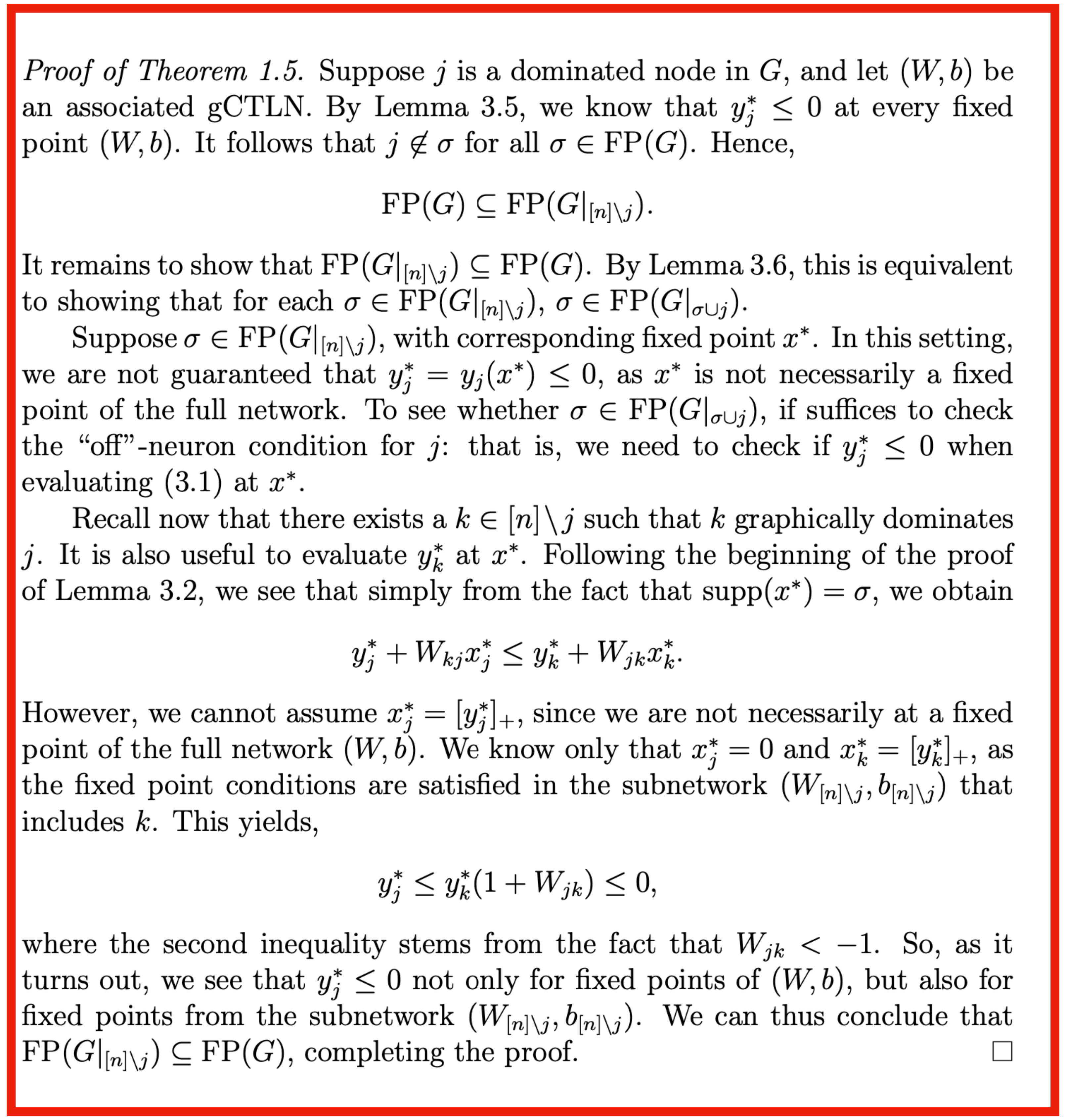
Proof of Theorem 1



Proof of Theorem 1.5. Suppose j is a dominated node in G, and let (W, b) be an associated gCTLN. By Lemma 3.5, we know that $y_j^* \leq 0$ at every fixed point (W, b). It follows that $j \notin \sigma$ for all $\sigma \in \text{FP}(G)$. Hence,

$$\text{FP}(G) \subseteq \text{FP}(G|_{[n]\setminus j}).$$

It remains to show that $\text{FP}(G|_{[n]\setminus j}) \subseteq \text{FP}(G)$. By Lemma 3.6, this is equivalent to showing that for each $\sigma \in \text{FP}(G|_{[n]\setminus j})$, $\sigma \in \text{FP}(G|_{\sigma \cup j})$.

Suppose $\sigma \in \text{FP}(G|_{[n]\setminus j})$, with corresponding fixed point x^*. In this setting, we are not guaranteed that $y_j^* = y_j(x^*) \leq 0$, as x^* is not necessarily a fixed point of the full network. To see whether $\sigma \in \text{FP}(G|_{\sigma \cup j})$, if suffices to check the "off"-neuron condition for j: that is, we need to check if $y_j^* \leq 0$ when evaluating (3.1) at x^*.

Recall now that there exists a $k \in [n] \setminus j$ such that k graphically dominates j. It is also useful to evaluate y_k^* at x^*. Following the beginning of the proof of Lemma 3.2, we see that simply from the fact that $\text{supp}(x^*) = \sigma$, we obtain

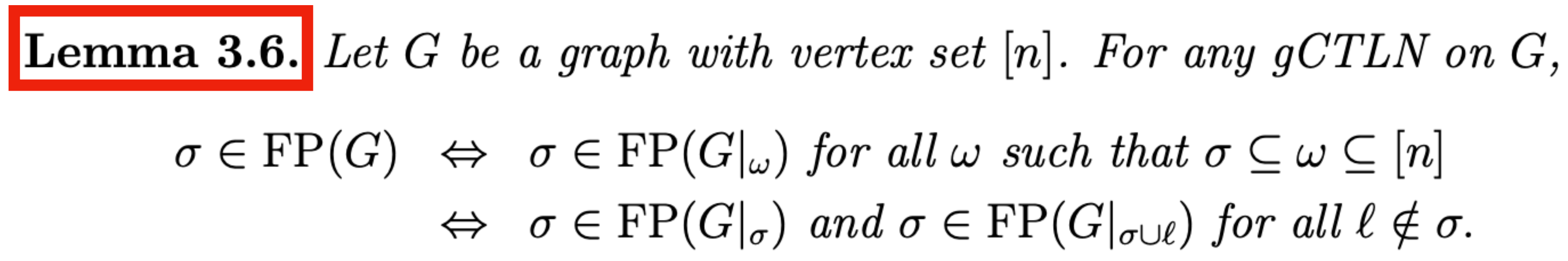
$$y_j^* + W_{kj}x_j^* \leq y_k^* + W_{jk}x_k^*.$$

However, we cannot assume $x_j^* = [y_j^*]_+$, since we are not necessarily at a fixed point of the full network (W, b). We know only that $x_j^* = 0$ and $x_k^* = [y_k^*]_+$, as the fixed point conditions are satisfied in the subnetwork $(W_{[n]\setminus j}, b_{[n]\setminus j})$ that includes k. This yields,

$$y_j^* \leq y_k^*(1 + W_{jk}) \leq 0,$$

where the second inequality stems from the fact that $W_{jk} < -1$. So, as it turns out, we see that $y_j^* \leq 0$ not only for fixed points of (W, b), but also for fixed points from the subnetwork $(W_{[n]\setminus j}, b_{[n]\setminus j})$. We can thus conclude that $\text{FP}(G|_{[n]\setminus j}) \subseteq \text{FP}(G)$, completing the proof. ☐

need some more lemmas...



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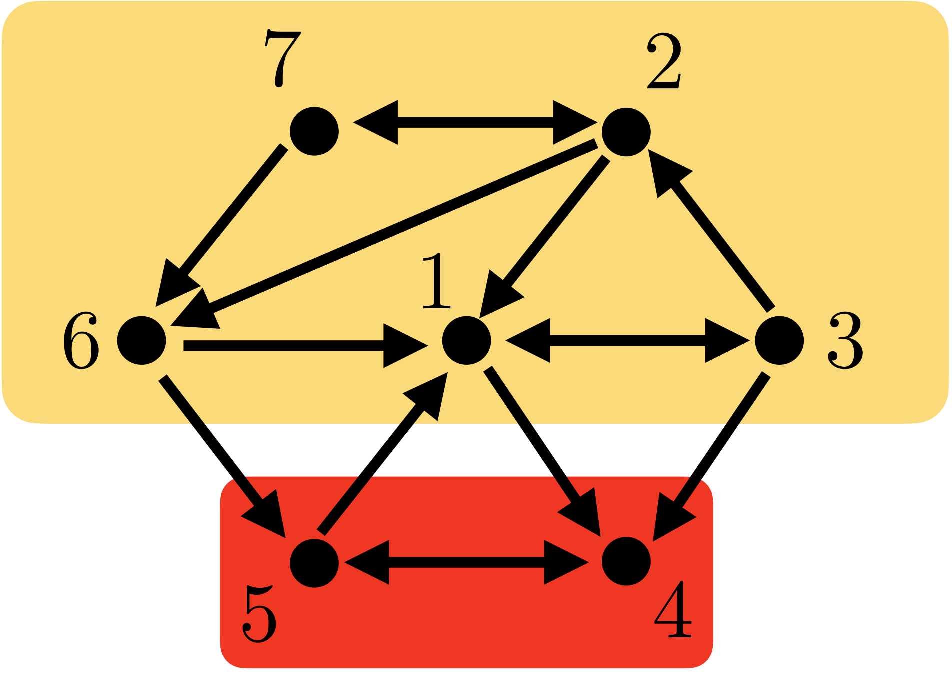
Plan of the talk

- Brief intro to TLNs, CTLNs, and gCTLNs
- Fixed points and attractors and graph rules
- Domination
- Dominoes and inhibitory control
- E-I TLNs
- Domination-reduction in connectomes

Dominoes!

<u>Conjecture</u>: network activity flows from any initial condition on the graph to the reduced network

G



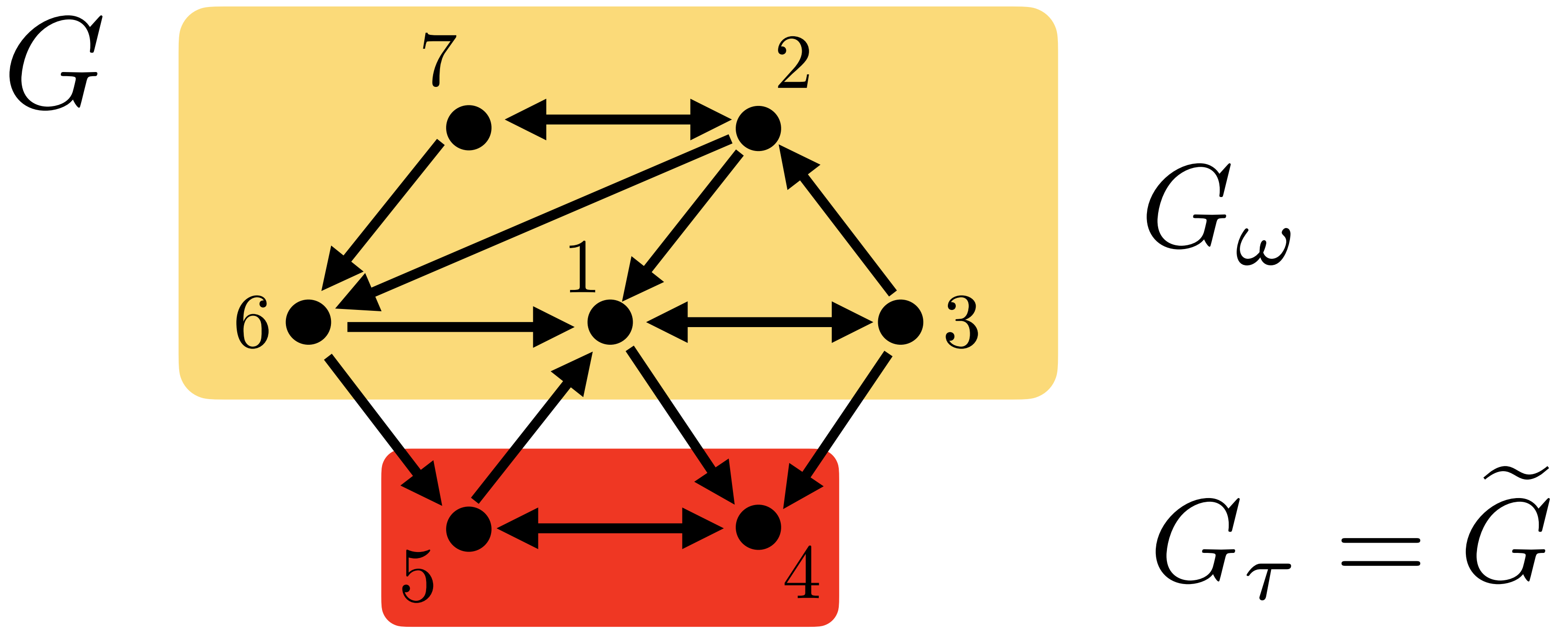
$\mathrm{FP(G)} = \{45\}$

G_ω

$G_\tau = \widetilde{G}$

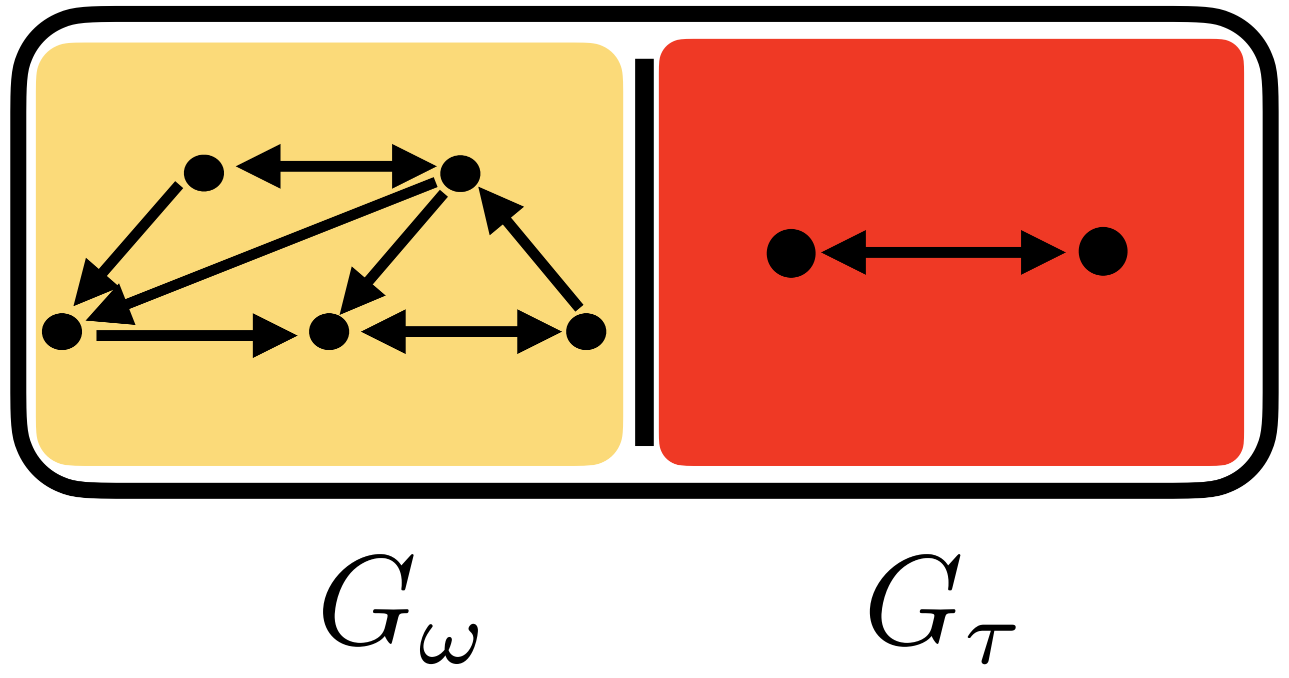
Dominoes!

Conjecture: network activity flows from any initial condition on the graph to the reduced network



$\mathrm{FP(G)} = \{45\}$

the "domino" of graph G



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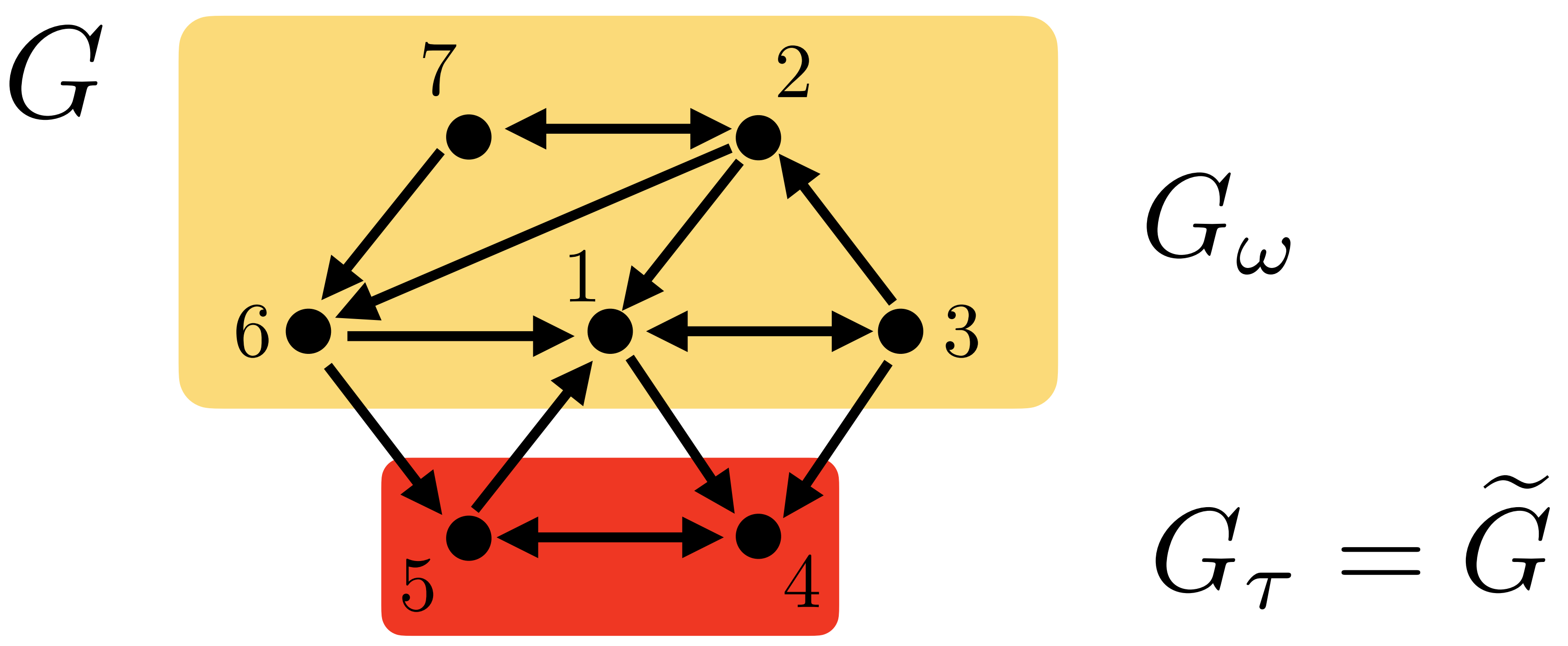

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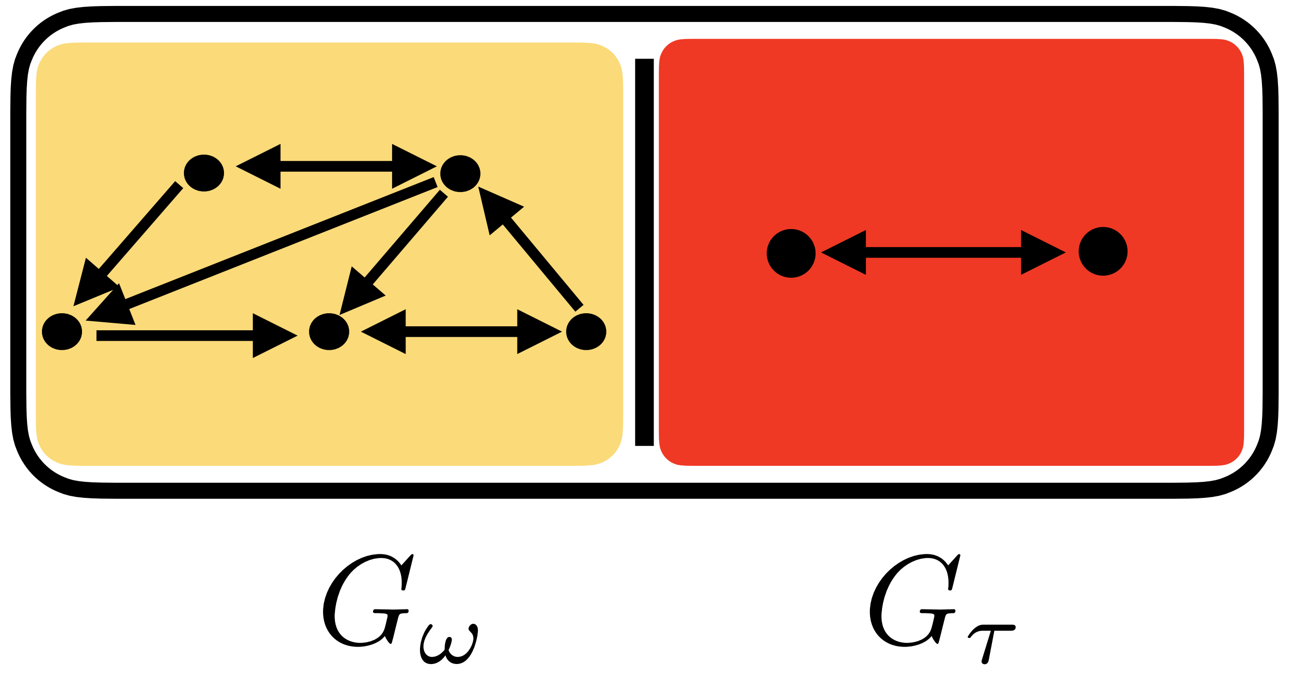

Fact (Thms 1 & 2): all the fixed points of G are supported in $G_\tau = \widetilde{G}$

Conjecture: network activity flows from $G_\omega \rightarrow G_\tau$

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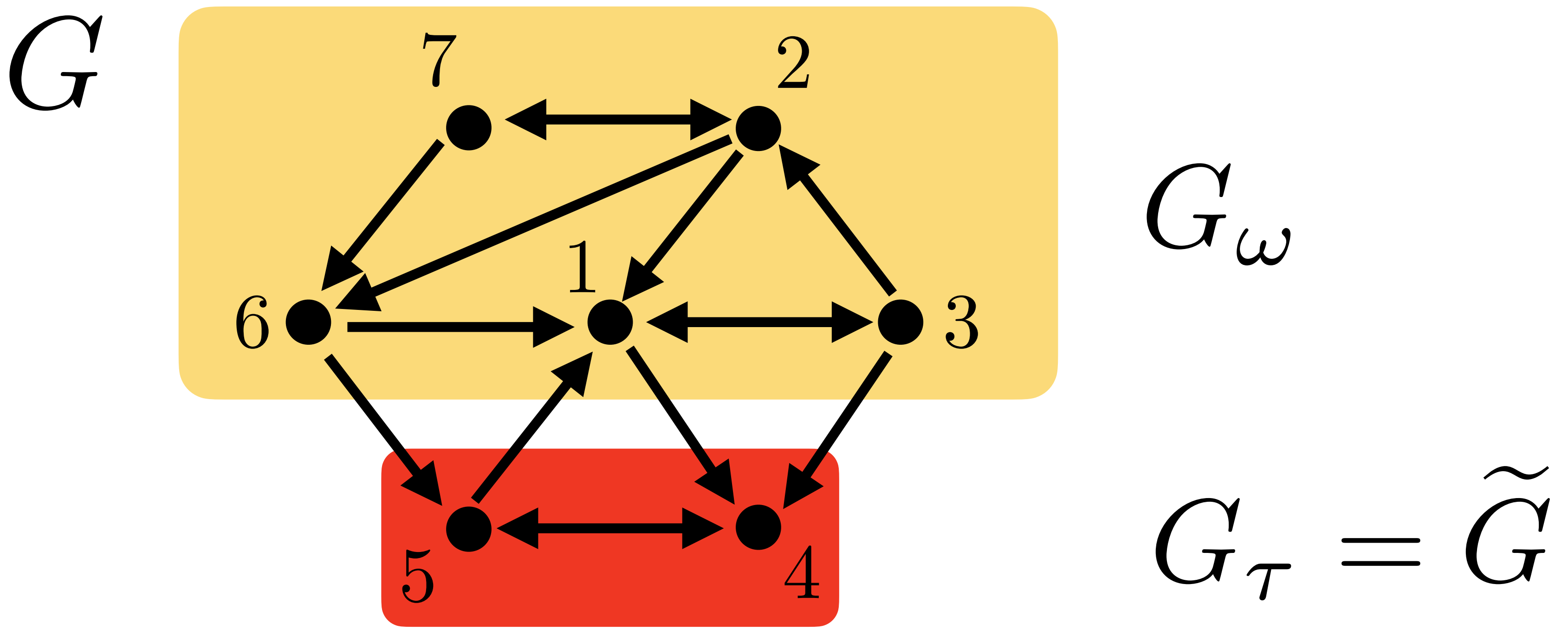

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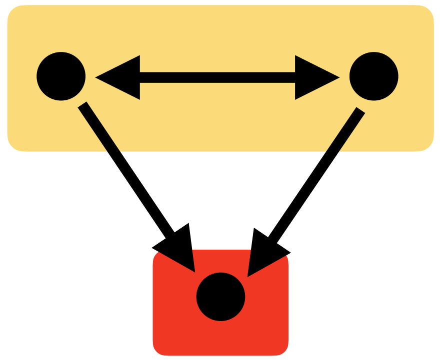

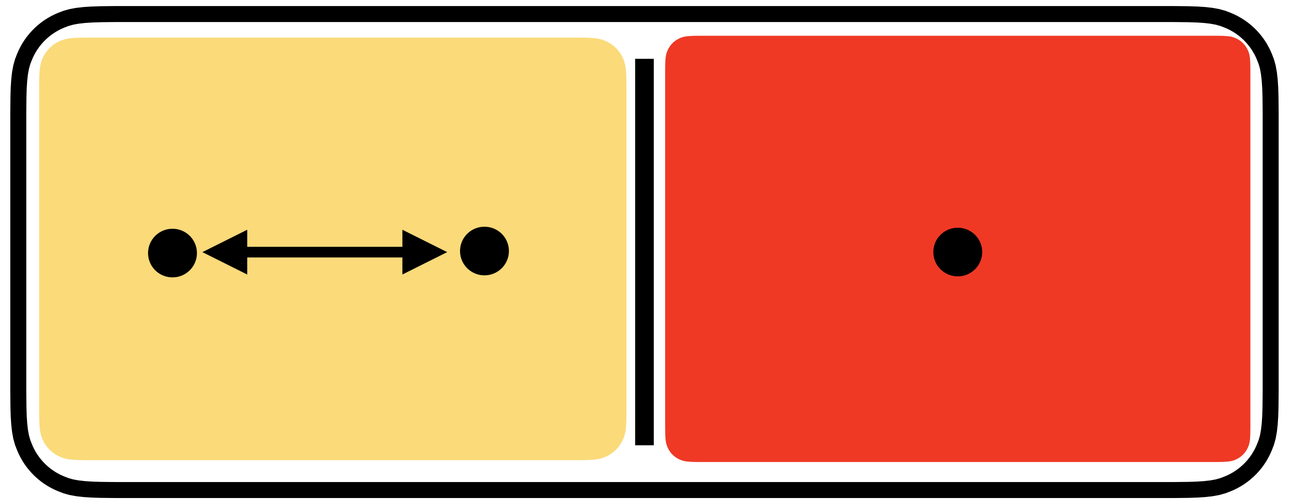

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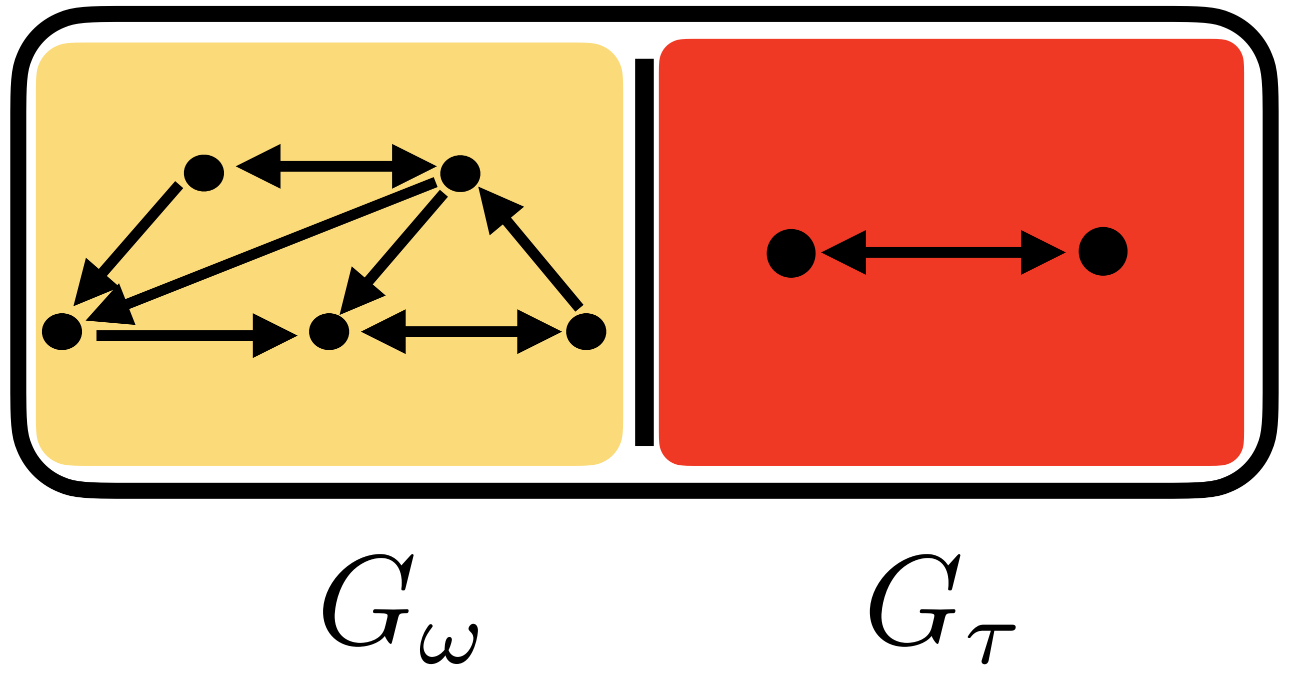

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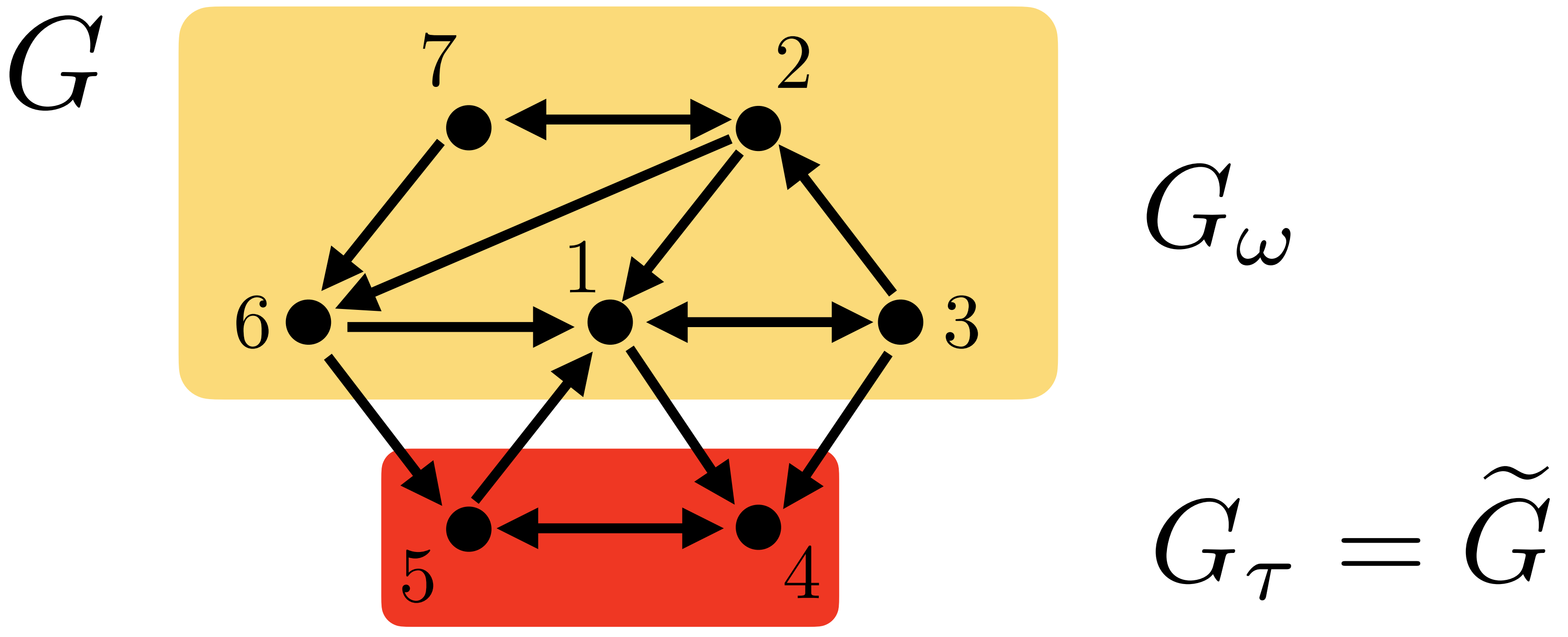

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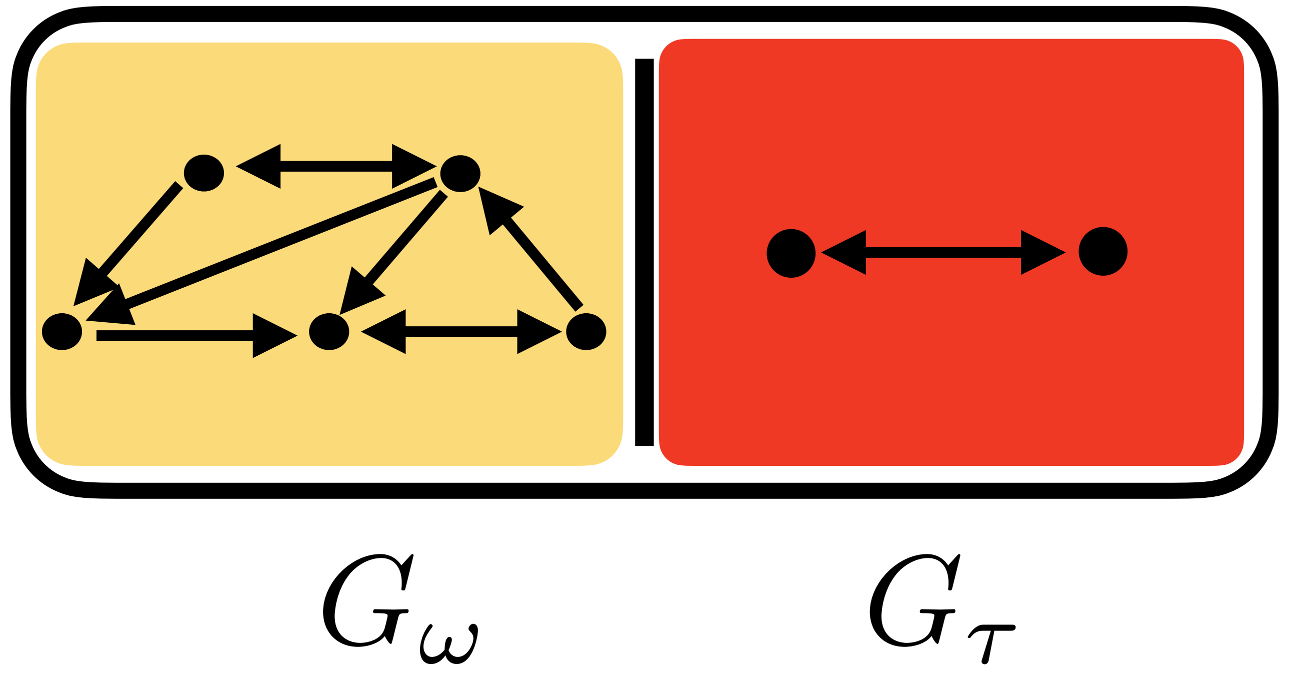

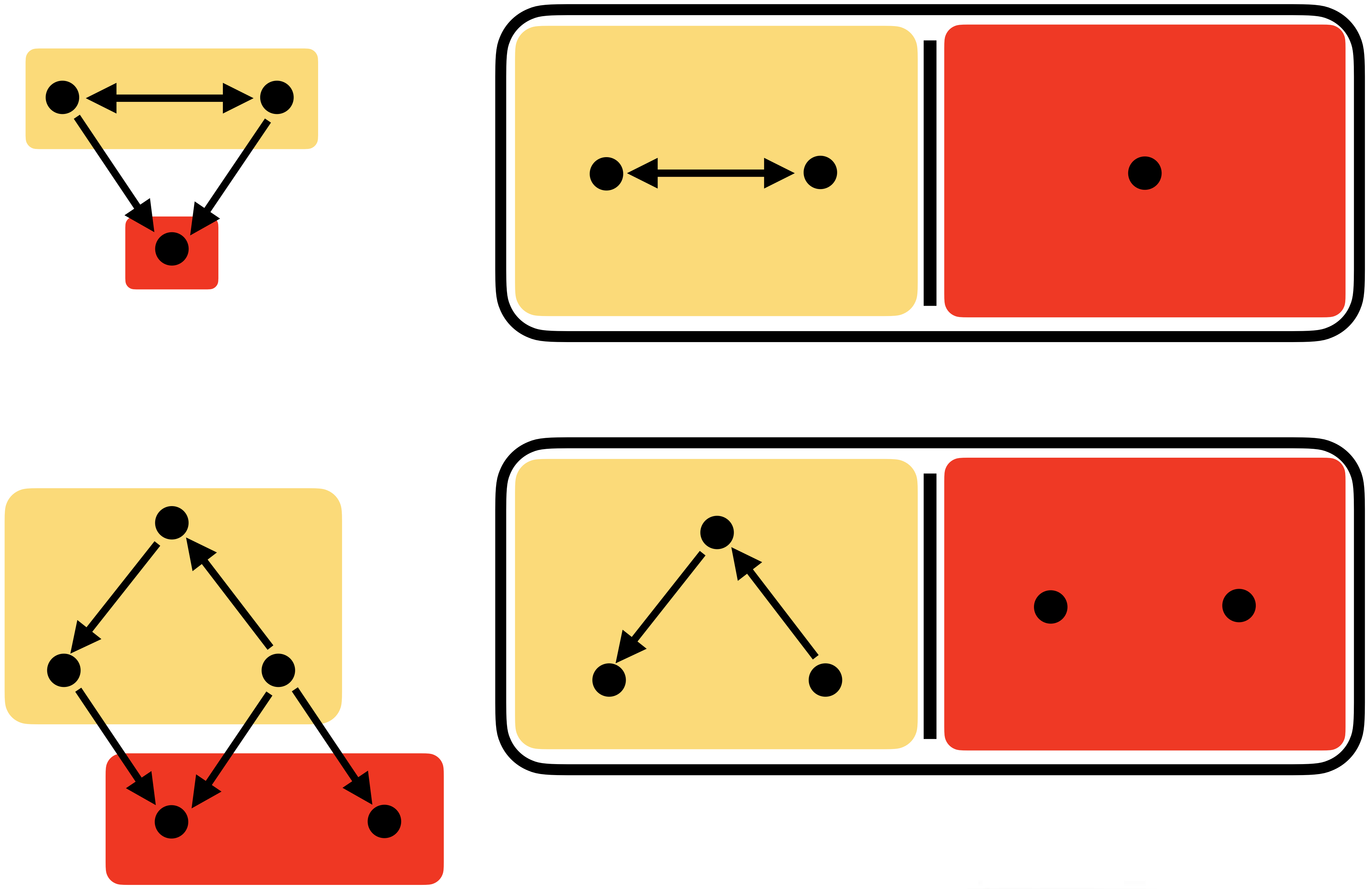

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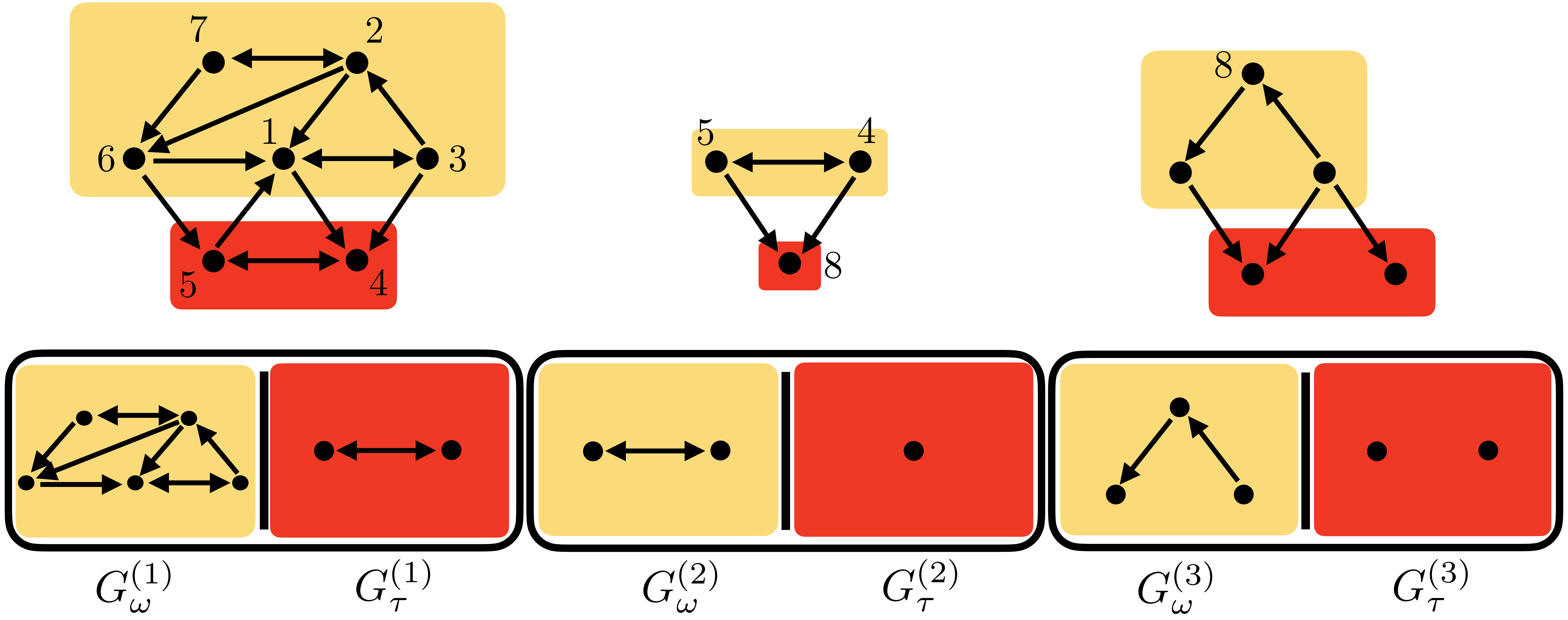
Dominoes! We can chain them together...



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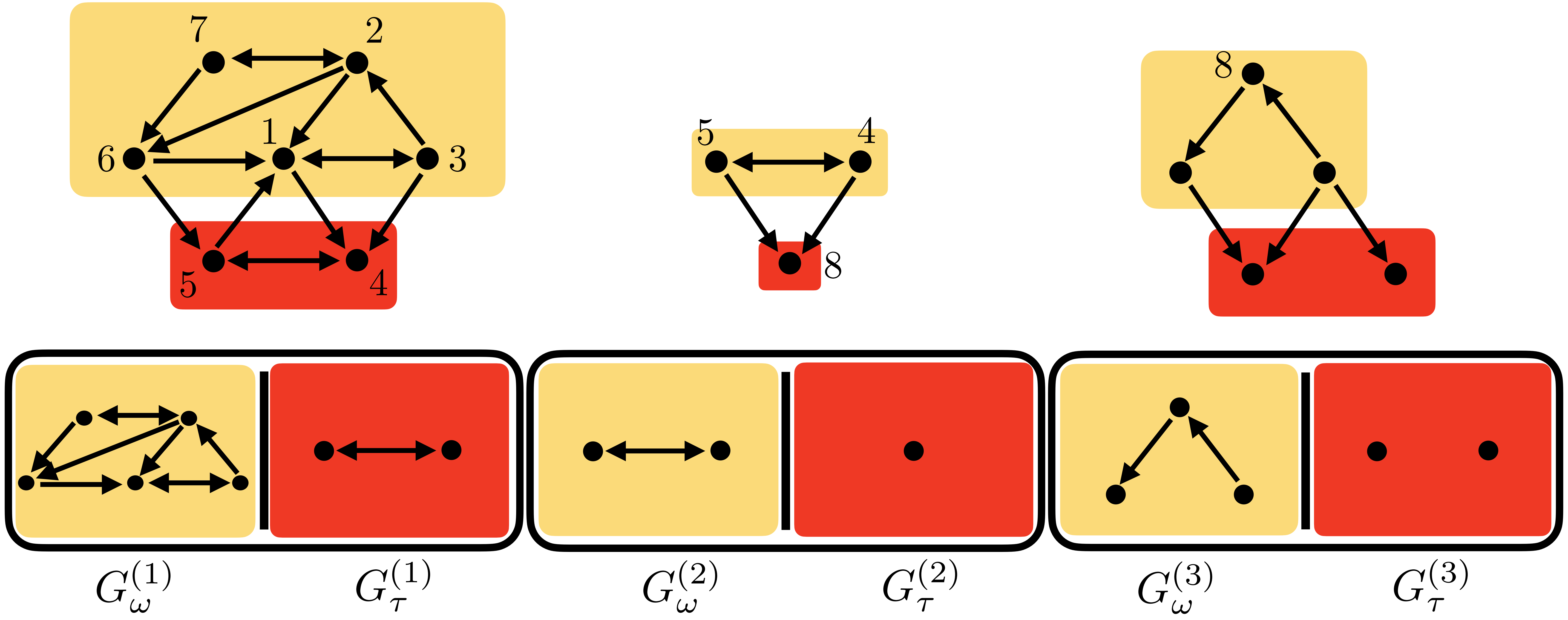

Theorem 3 (2024)

If we glue reducible graphs together along their dominoes, in a linear chain, so that G_τ of one is identified with a subgraph of G_ω of the next, then the glued graph reduces to the final $G_\tau^{(i)}$.

Curto 2024 (unpublished)

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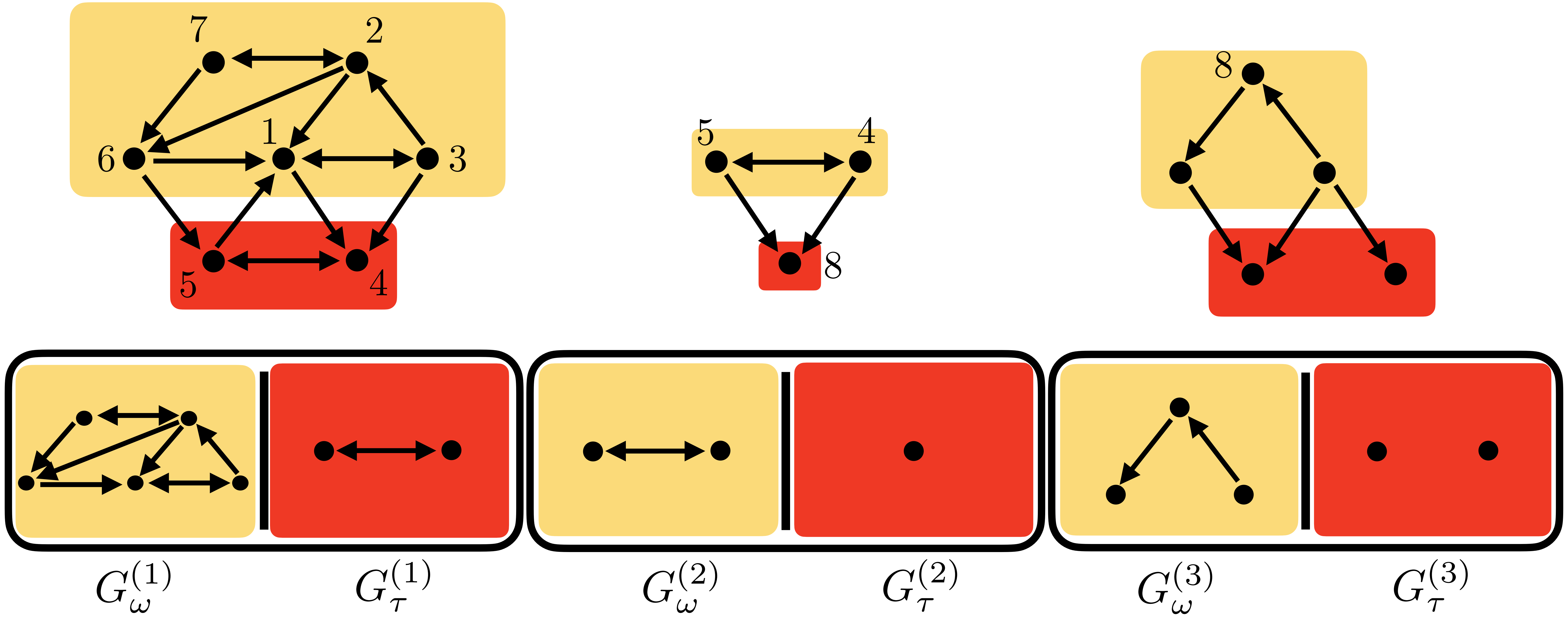
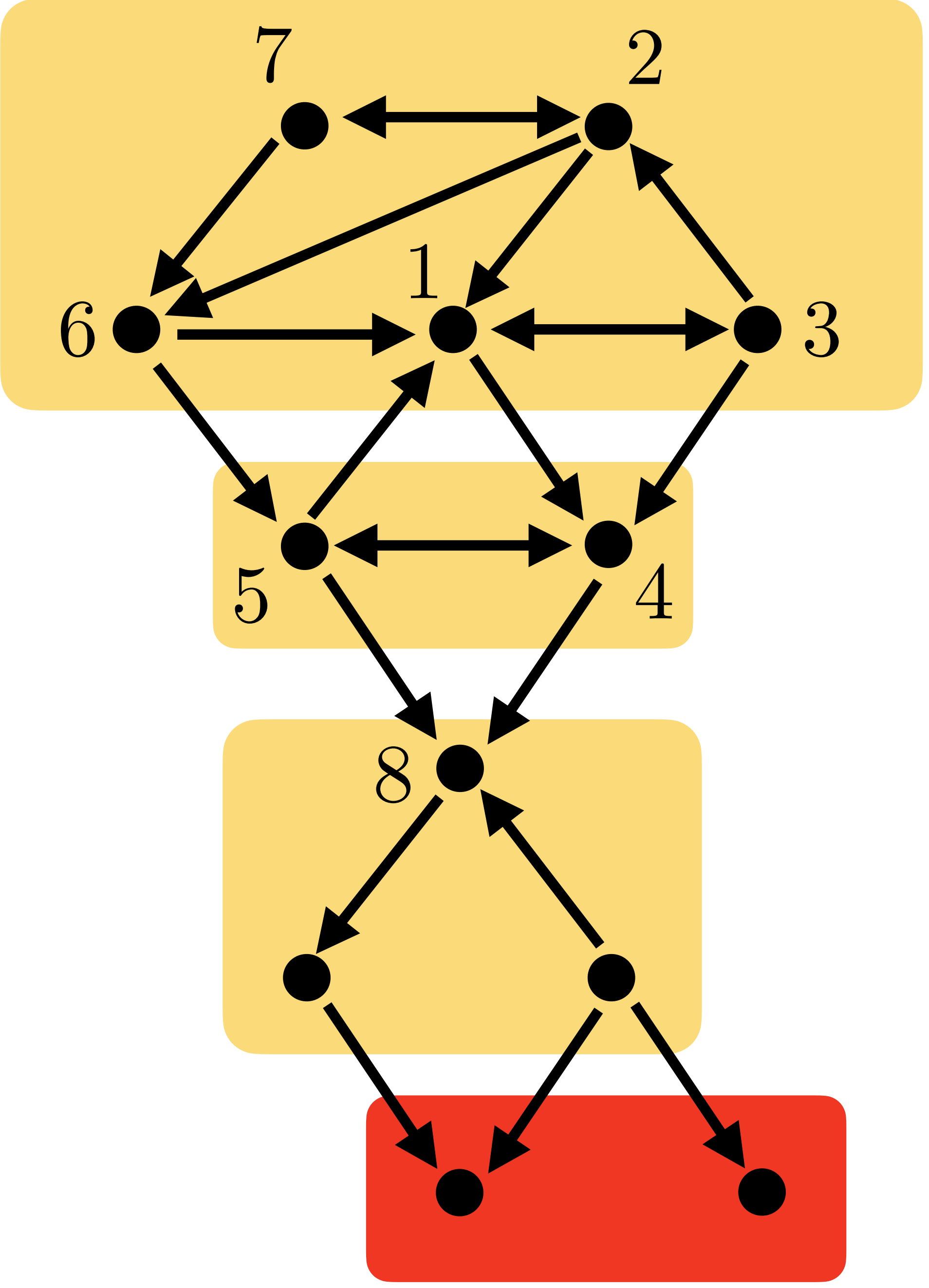


glued graph G



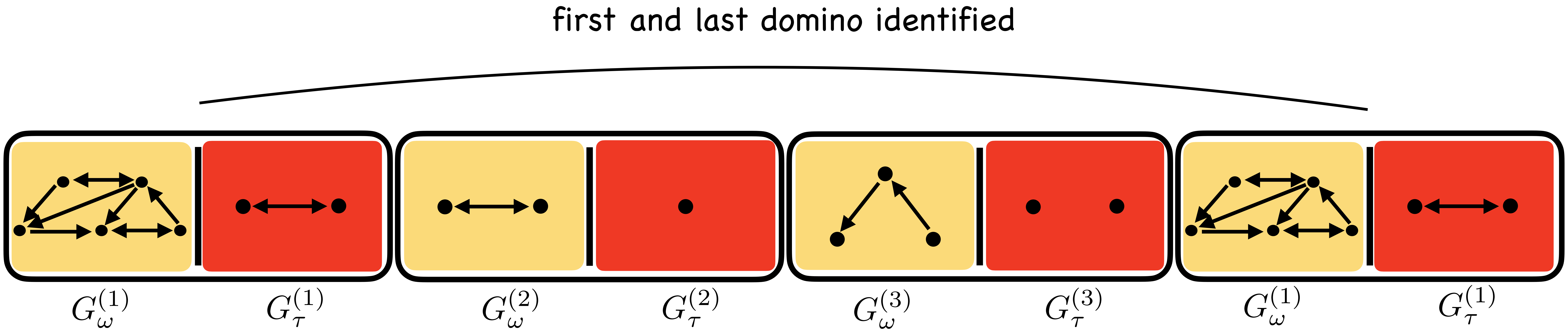
$$\widetilde{G} = G_\tau^{(3)}$$

$$\mathrm{FP}(G) = \mathrm{FP}(G_\tau^{(3)})$$

Theorem 3 (2024)
If we glue reducible graphs together along their dominoes, in a linear chain, so that G_τ of one is identified with a subgraph of G_ω of the next, then the glued graph reduces to the final $G_\tau^{(i)}$.

Curto 2024 (unpublished)

What about a cyclic chain?



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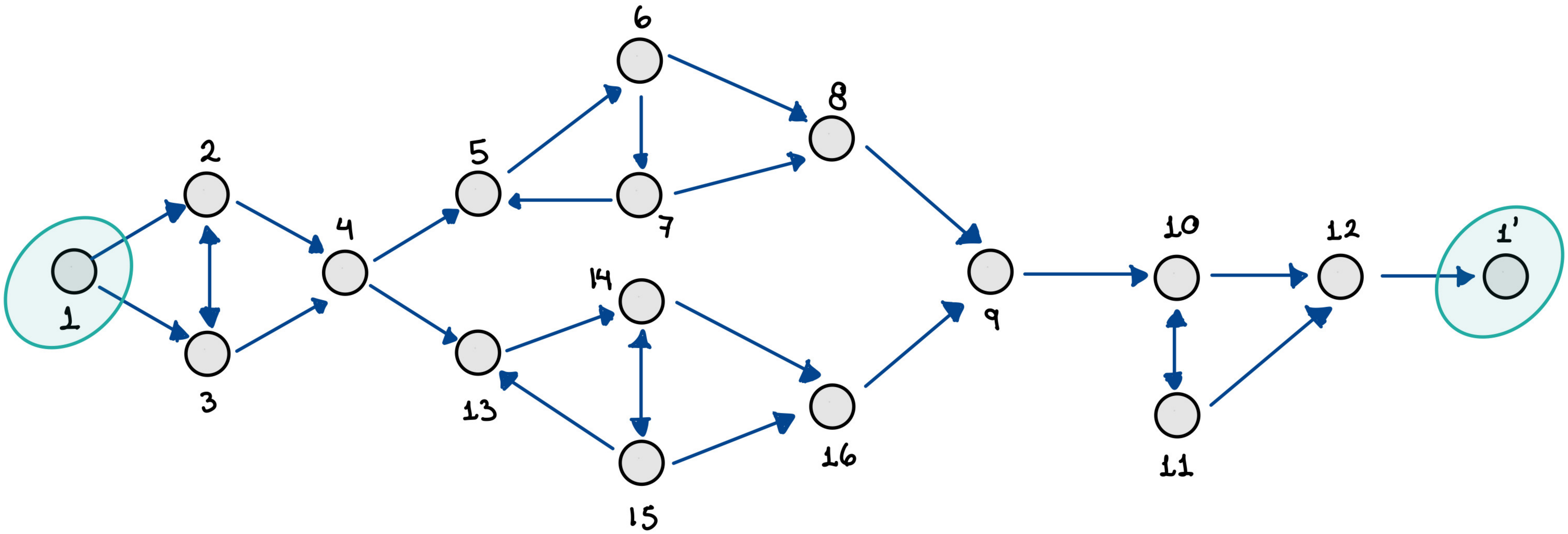
~~If we glue reducible graphs together along their dominoes, in a linear chain, so that G_τ of one is identified with a subgraph of G_ω of the next, then the glued graph reduces to the final $G_\tau^{(i)}$.~~

Cyclic chain example



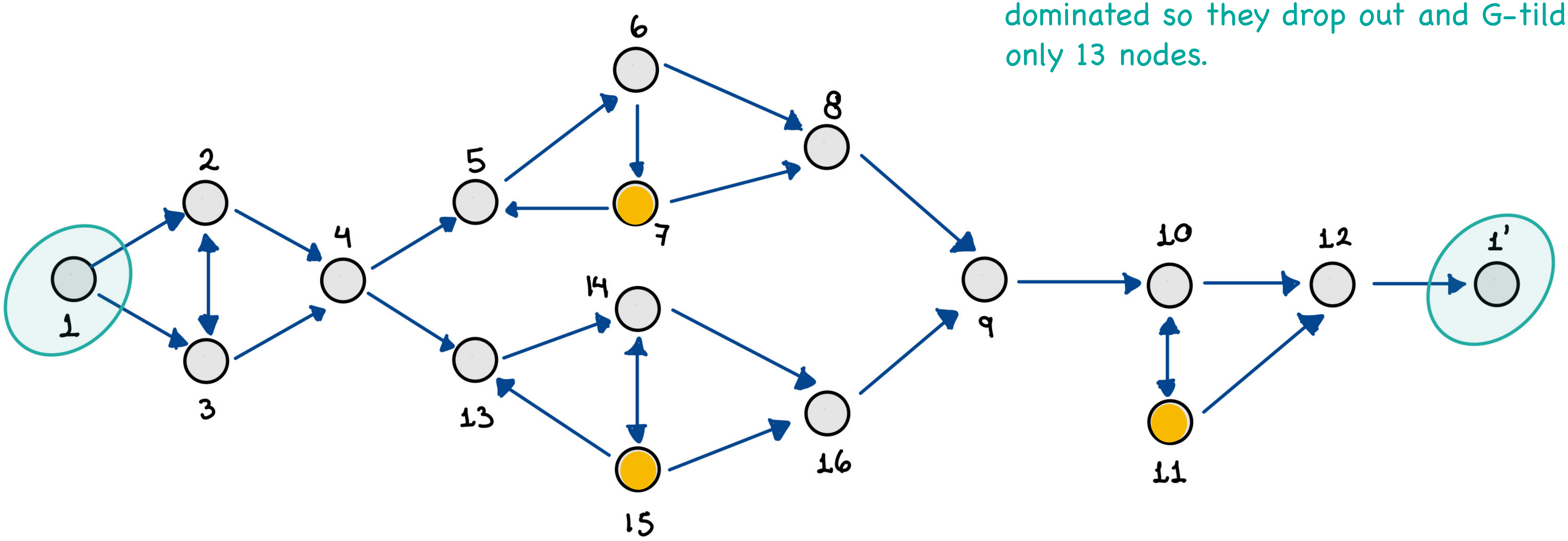
Identify $1 \equiv 1'$ at the end

Domination reduction cannot be done, and the network activity will loop around.

Cyclic chain example

Domination reductions:

1) Without identifying 1′ and 1, G reduces to 1′
2) After identifying 1′ and 1, nodes 7, 11, 15 are dominated so they drop out and G-tilde has only 13 nodes.



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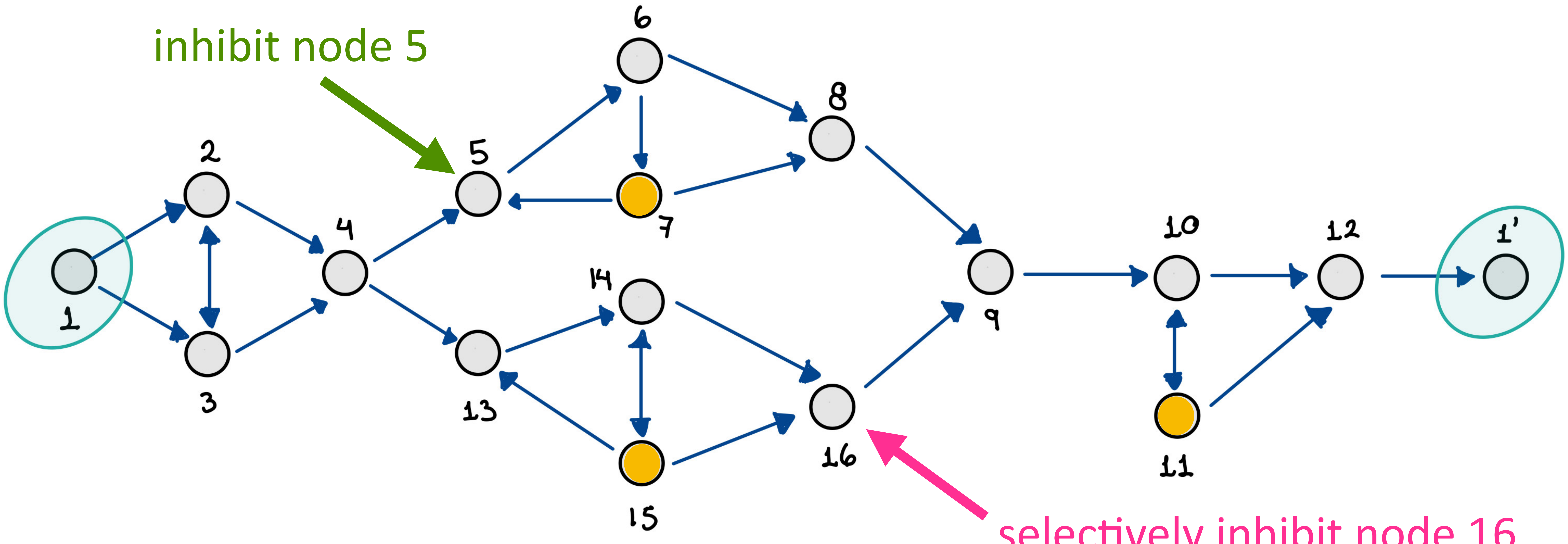
Inhibitory control

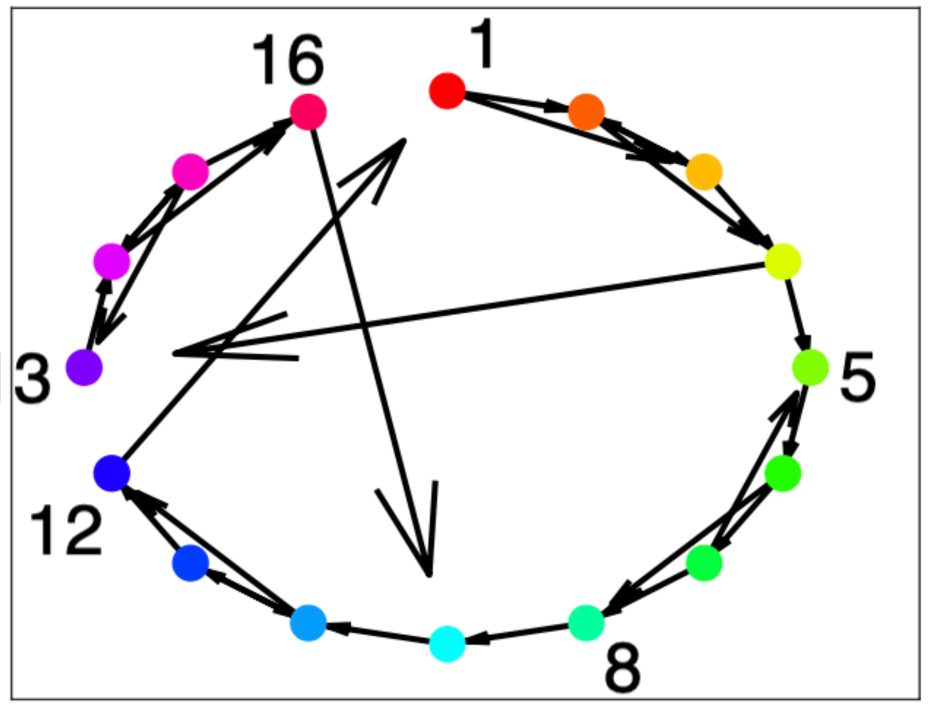


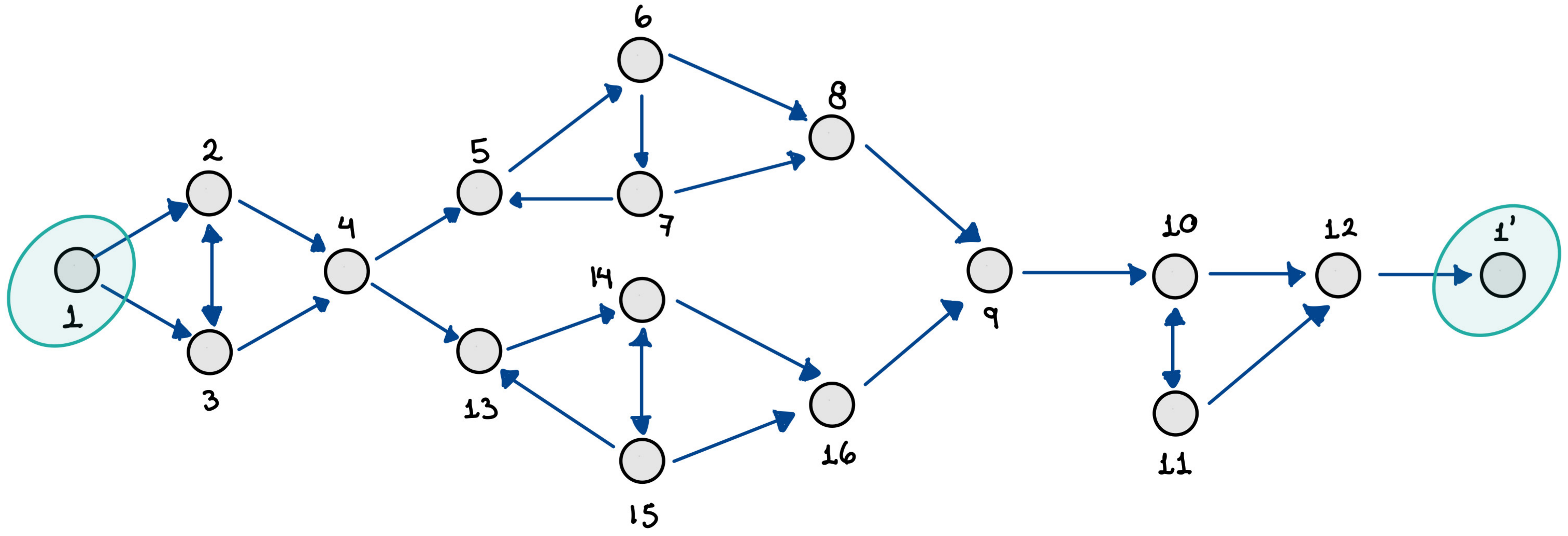
inhibit node 5

selectively inhibit node 16

Identify $1 \equiv 1'$ at the end

What if you selectively inhibit one of the neurons?

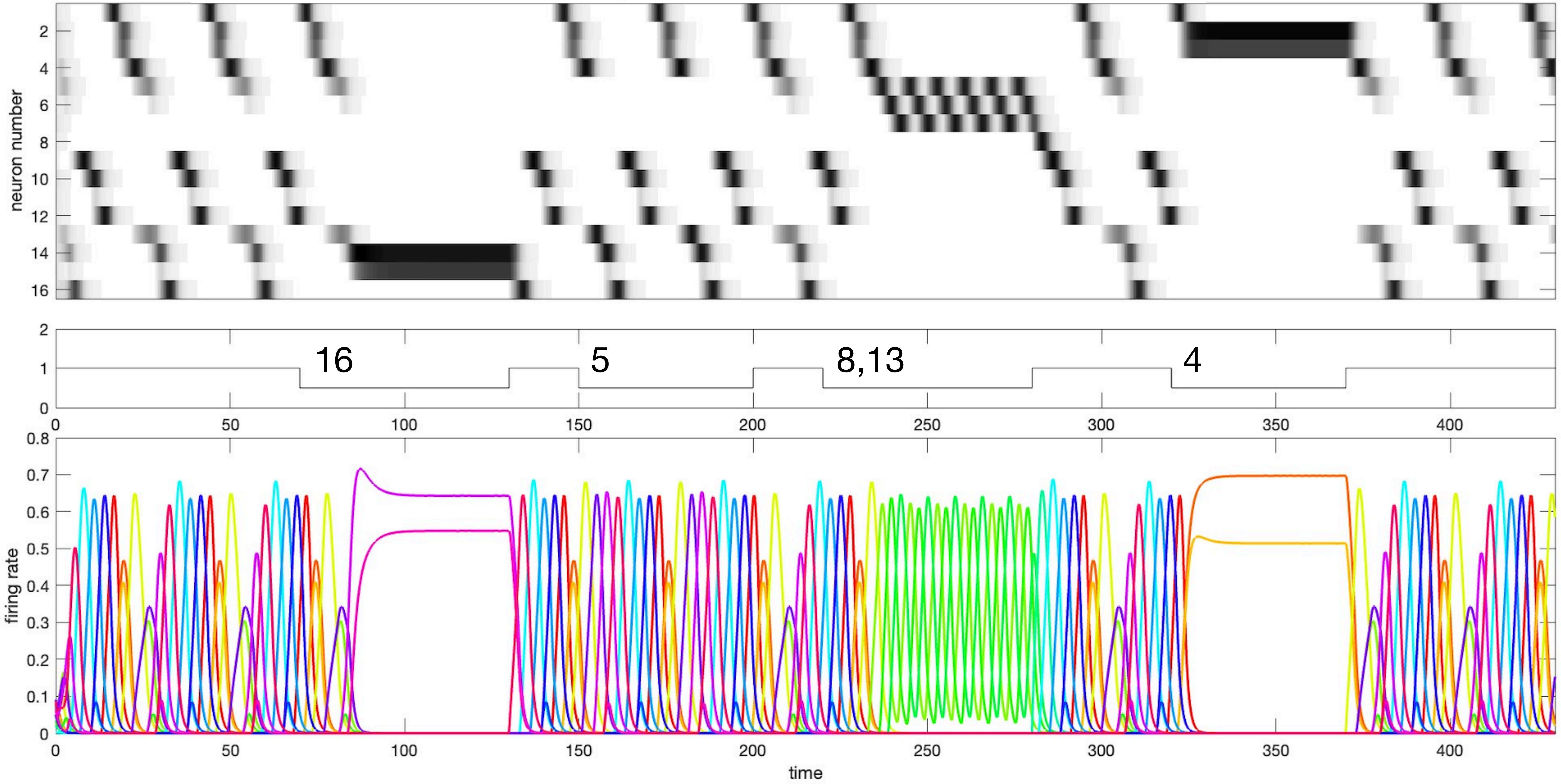




initial
"resting state"

Identify 1 ≡ 1' at the end

Control by inhibitory pulses:



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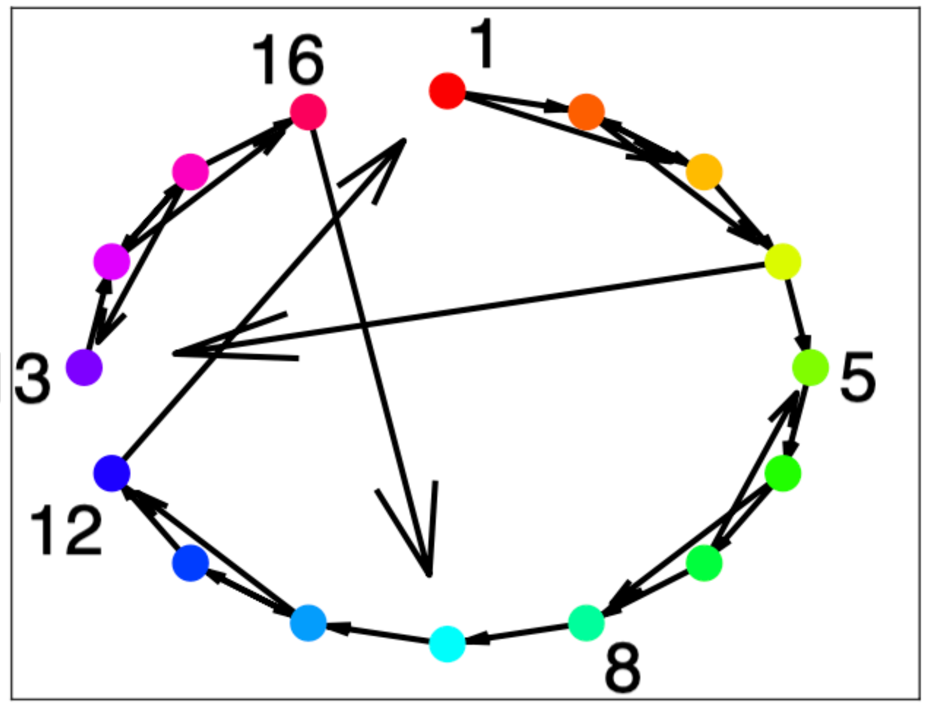

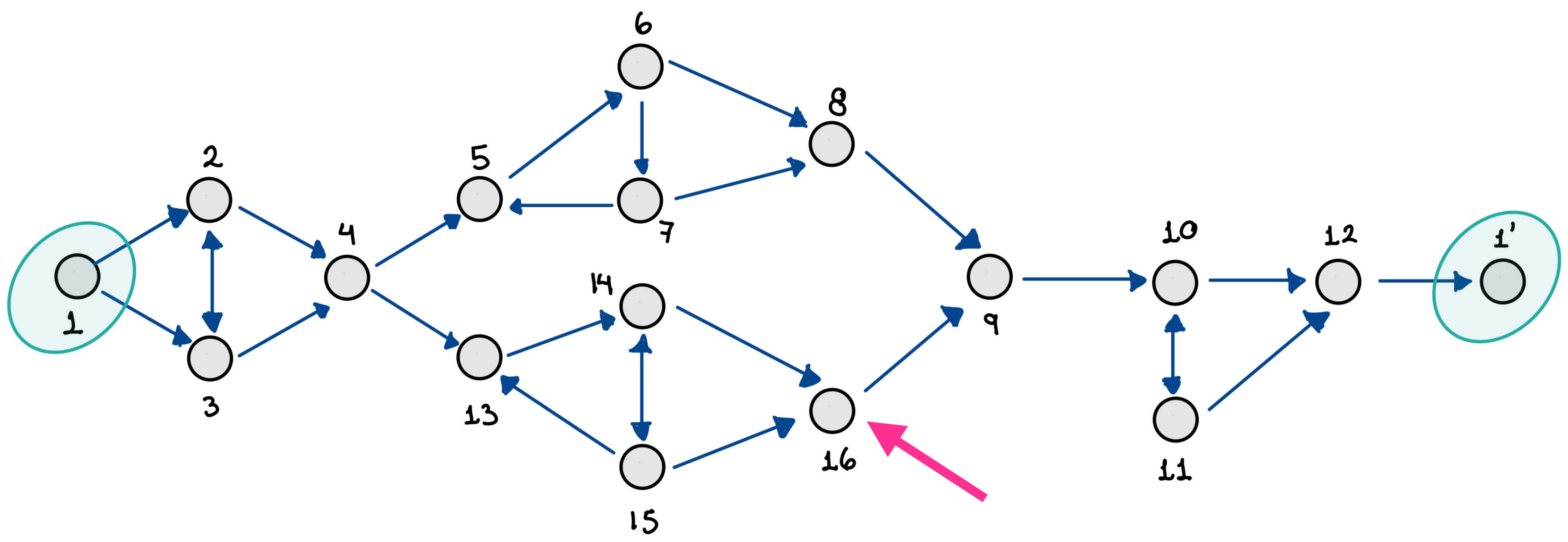

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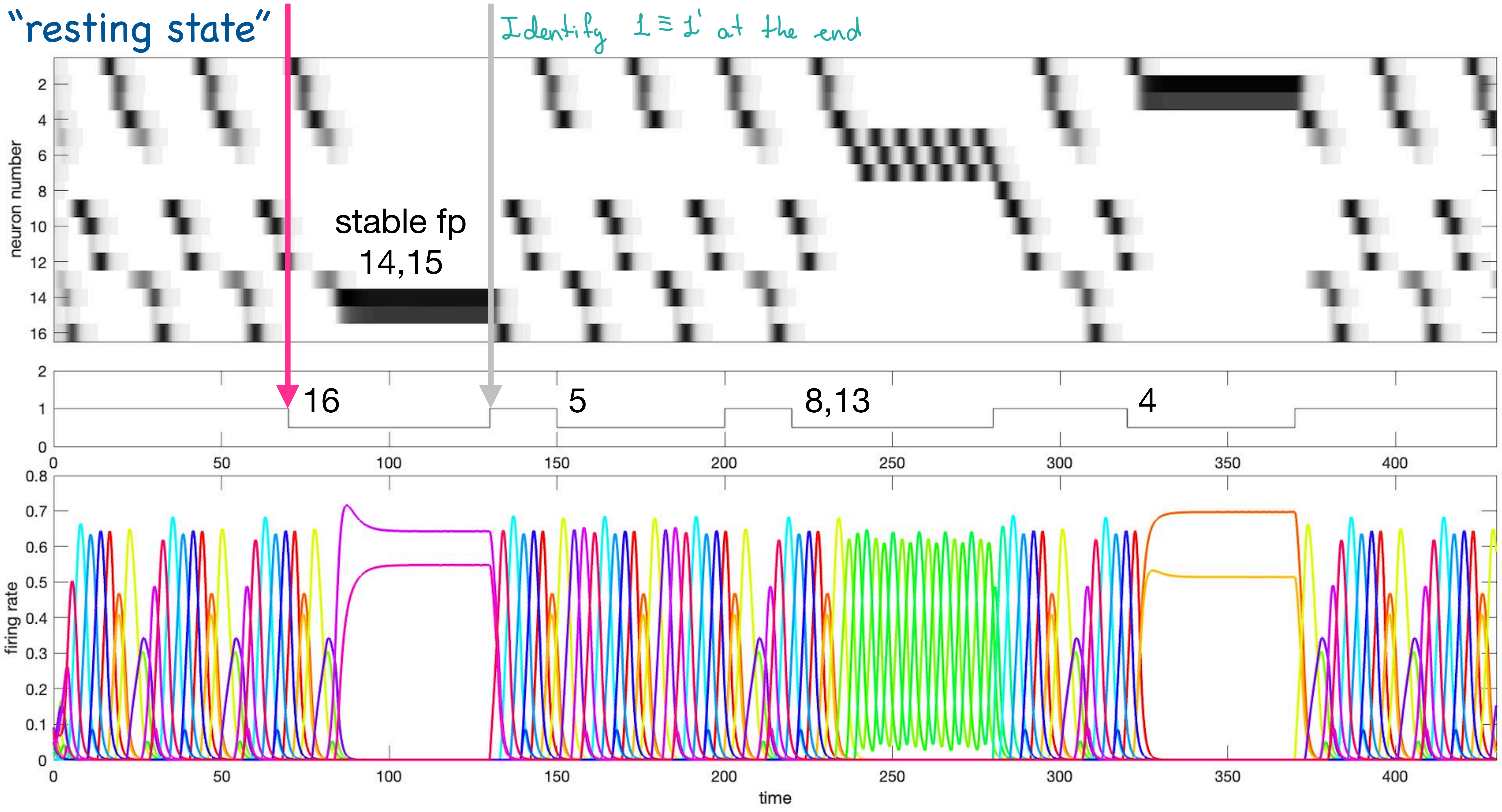

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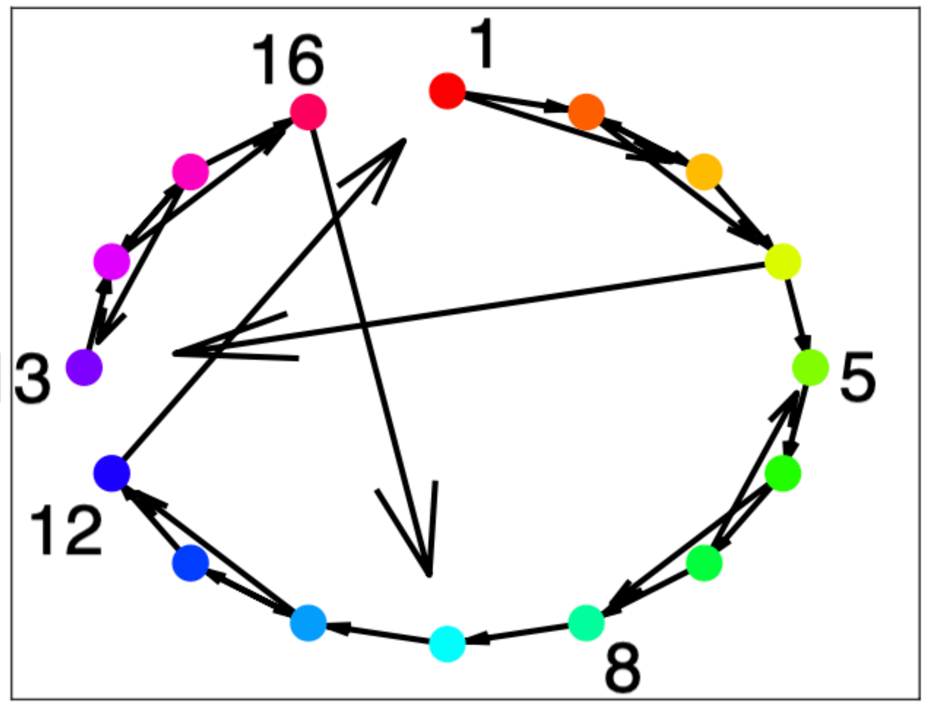

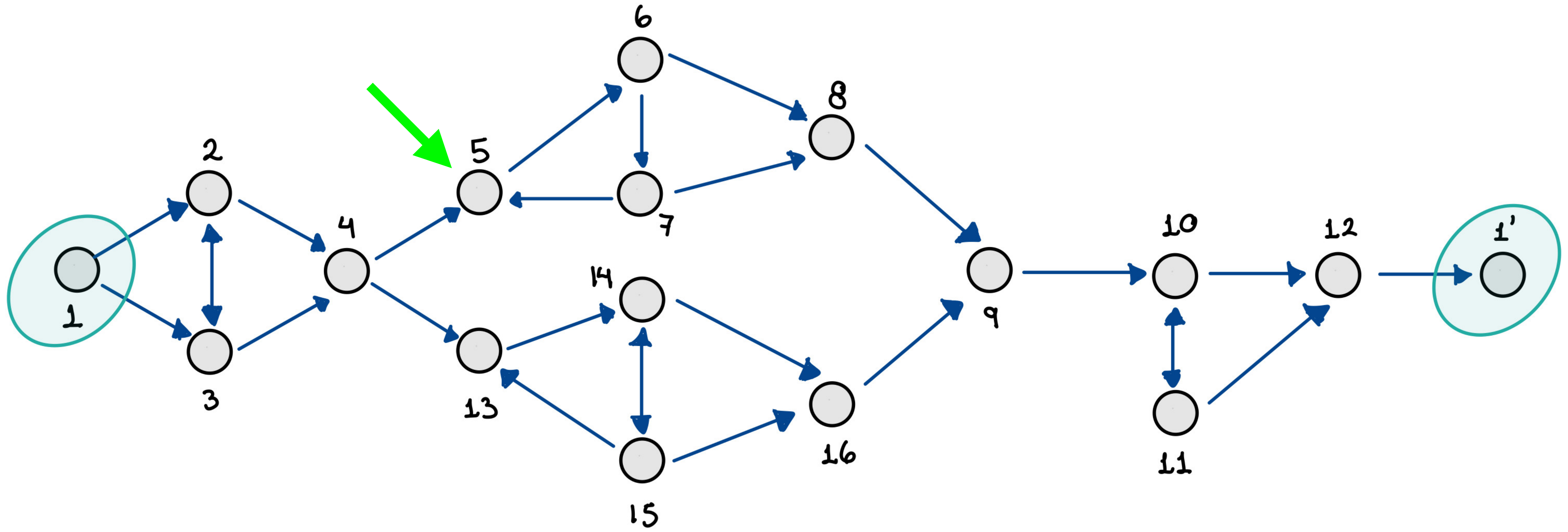

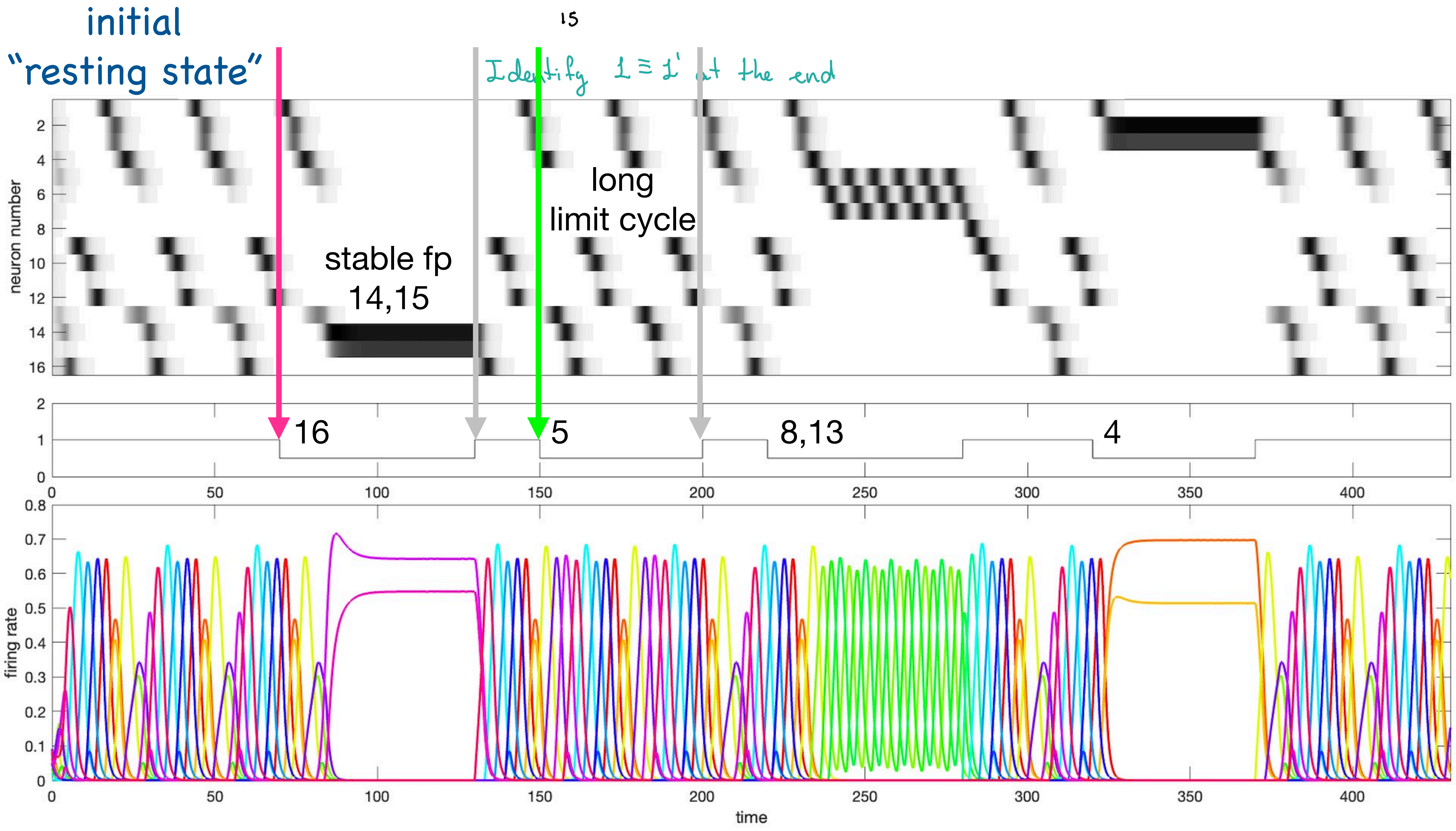

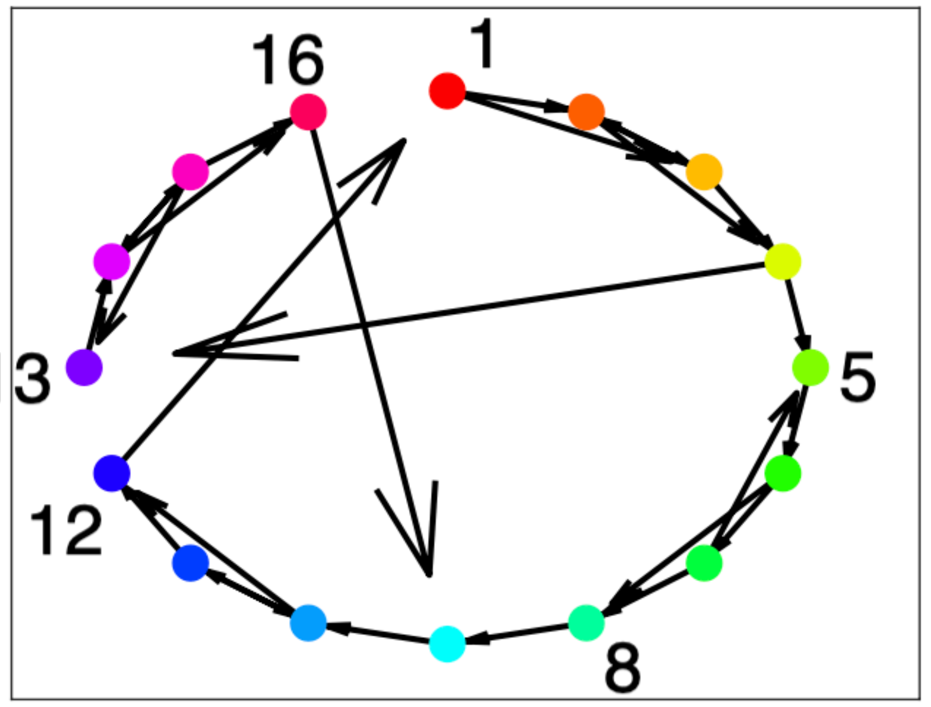

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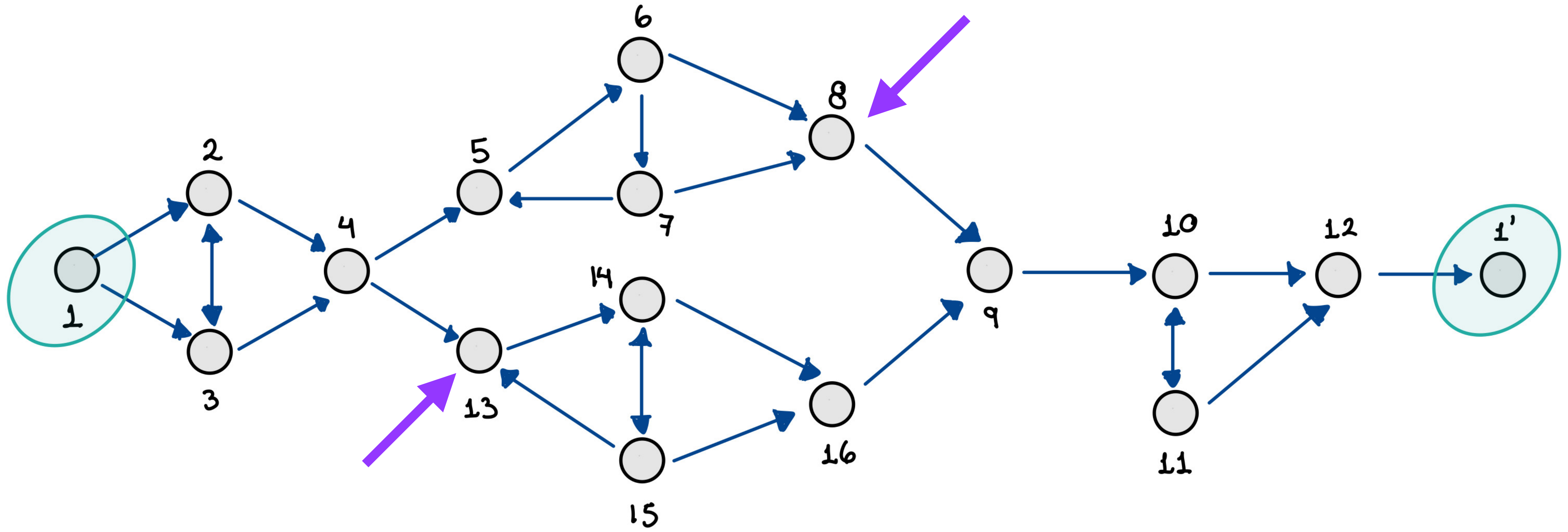
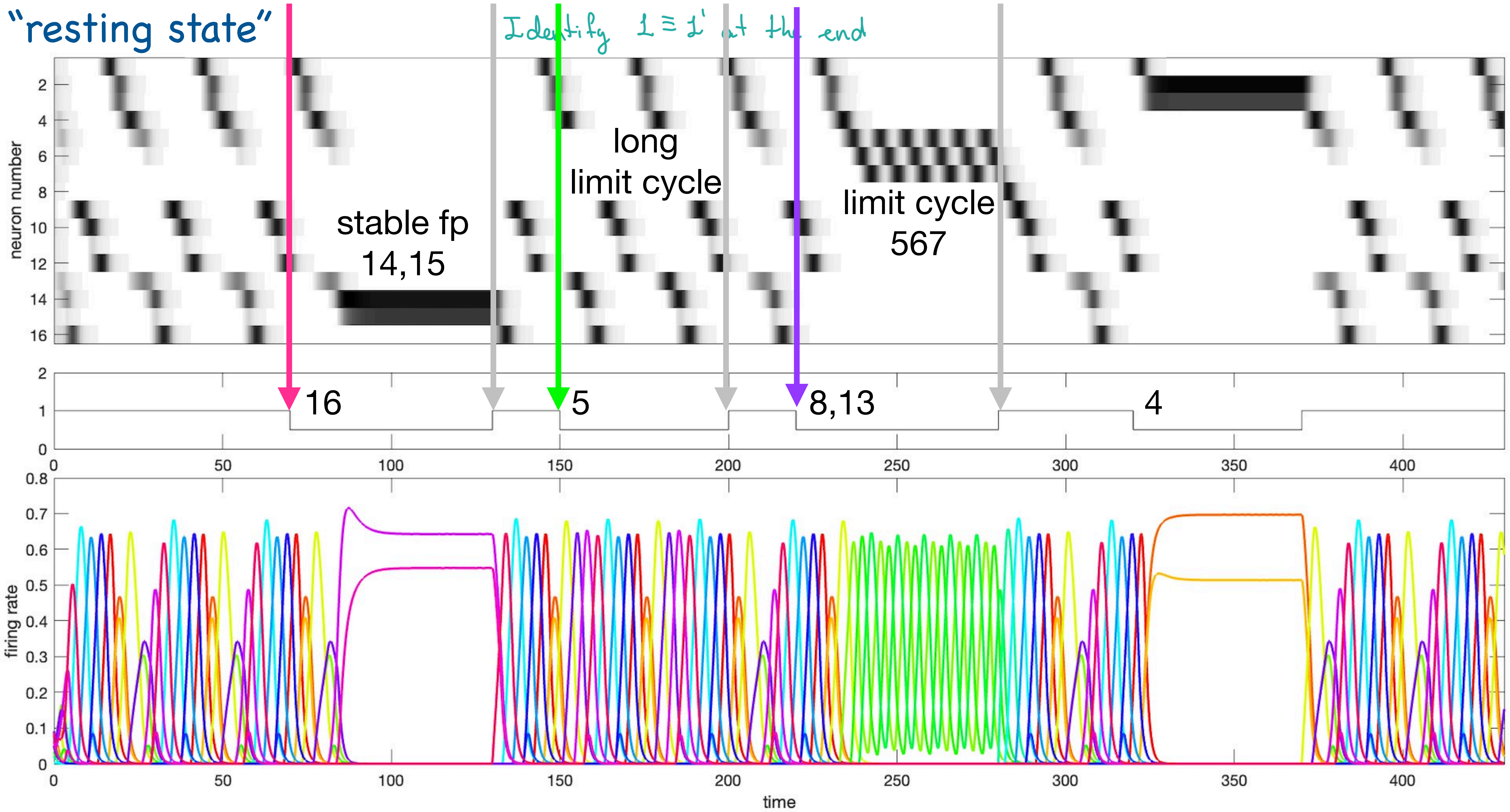


initial "resting state"



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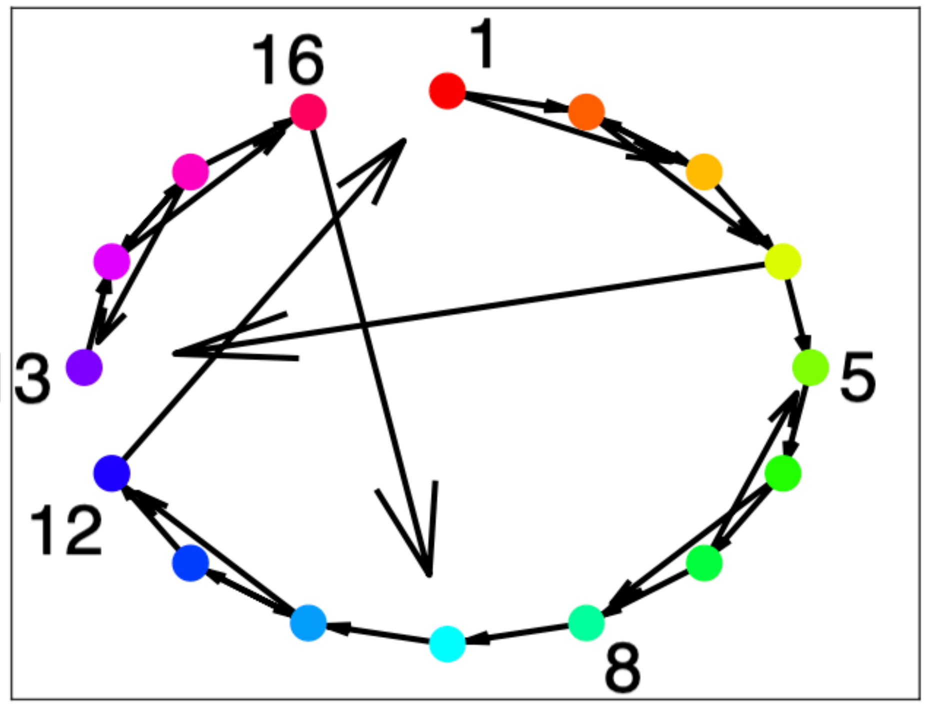

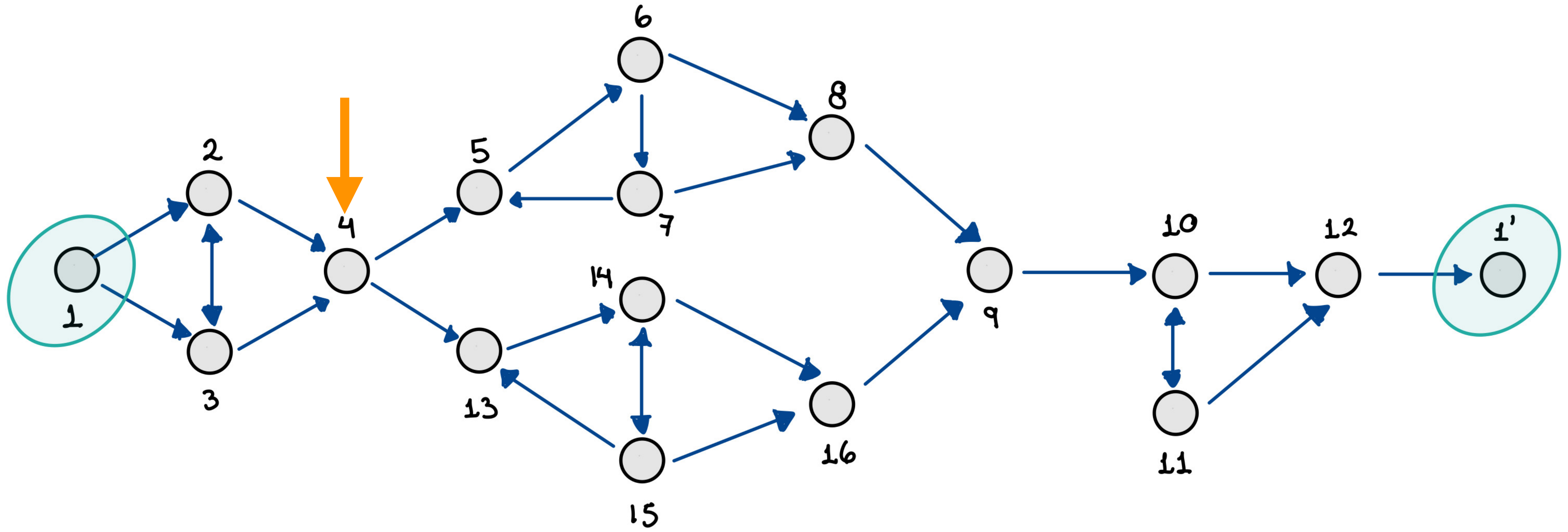

initial
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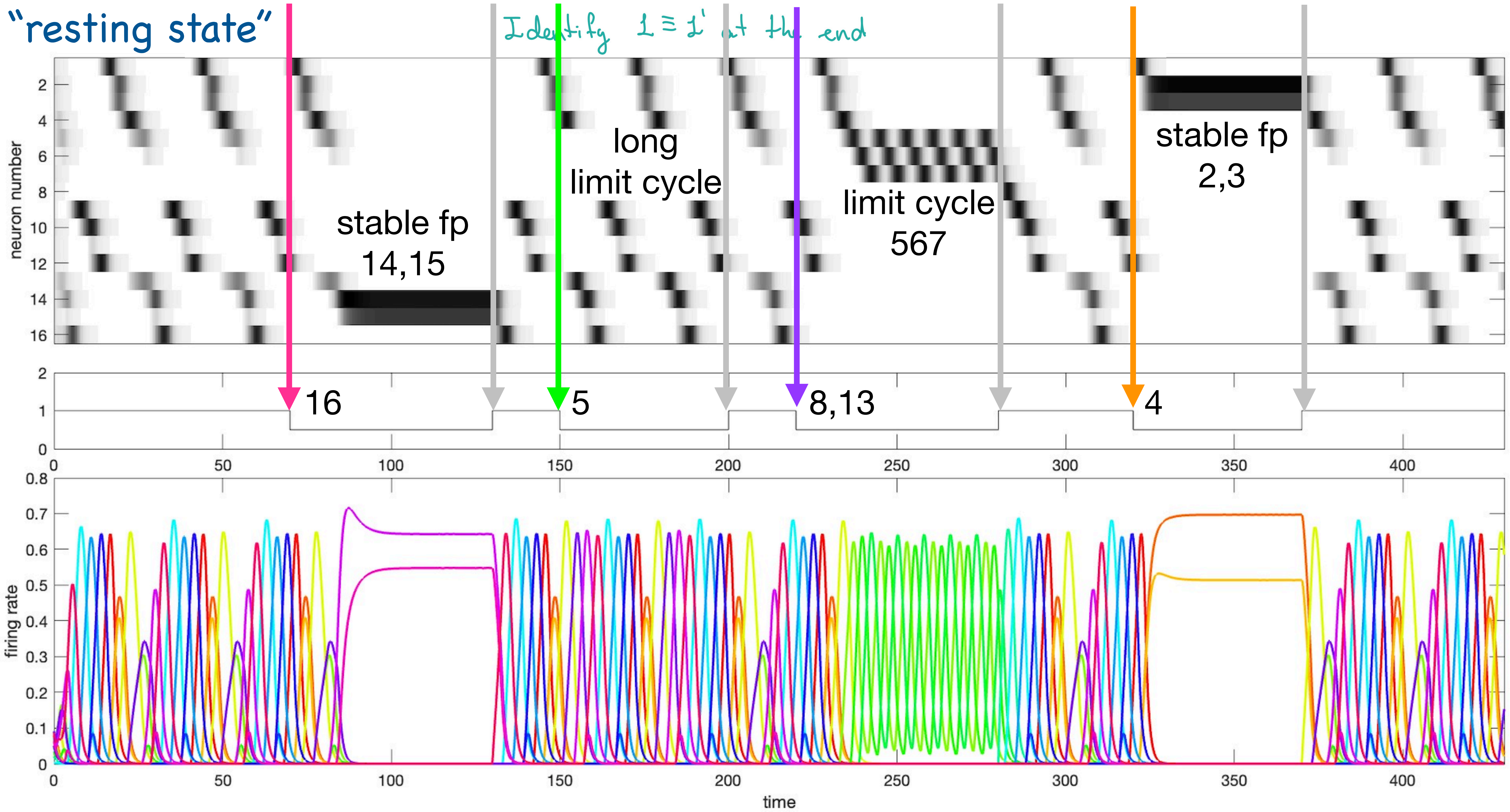
Identify 1 ≡ 1' at the end

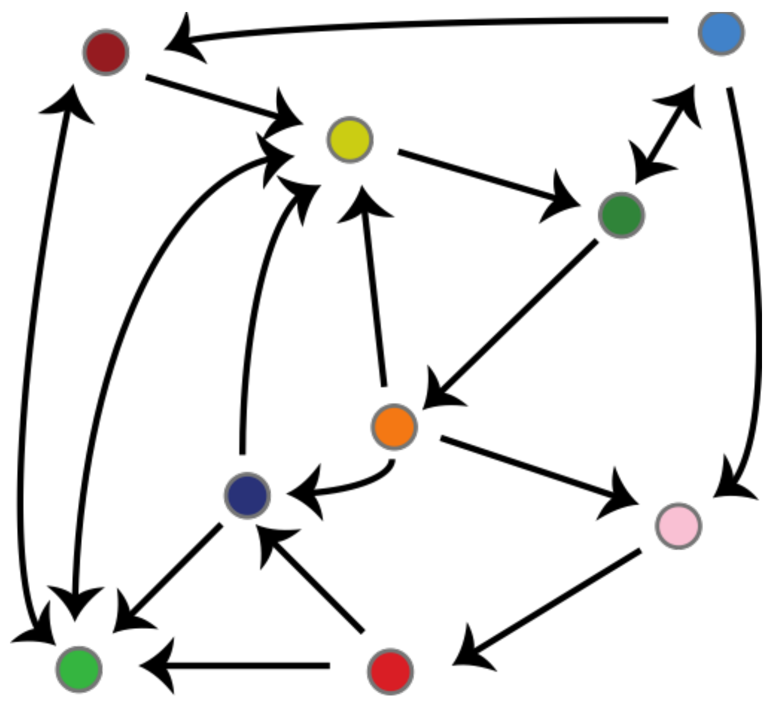


Plan of the talk

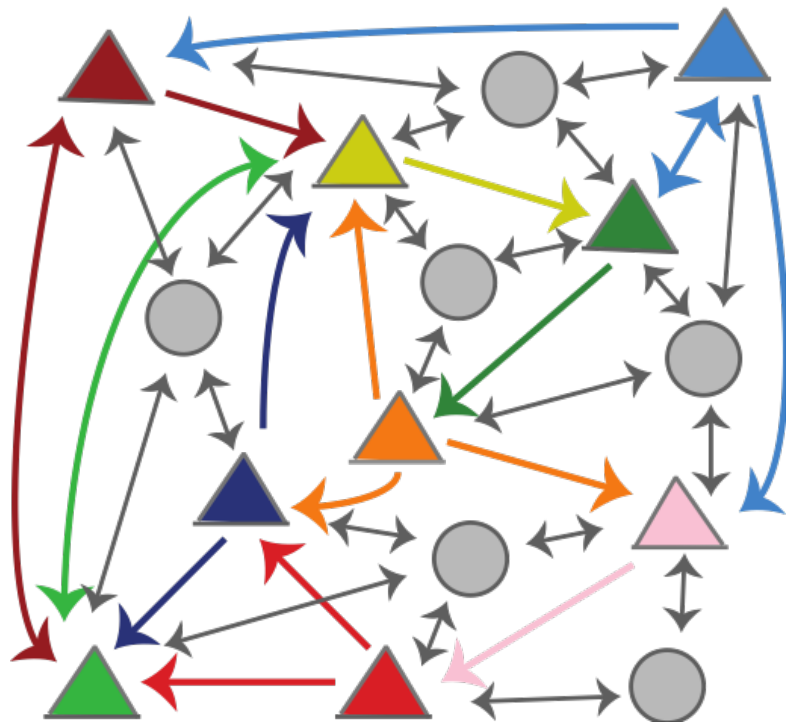
- Brief intro to TLNs, CTLNs, and gCTLNs
- Fixed points and attractors and graph rules
- Domination
- Dominoes and inhibitory control
- E-I TLNs
- Domination-reduction in connectomes

So far, everything we have done for CTLNs/gCTLNs has assumed negative (inhibitory) weights on the W matrix.

graph G



Idea: network of excitatory and inhibitory cells



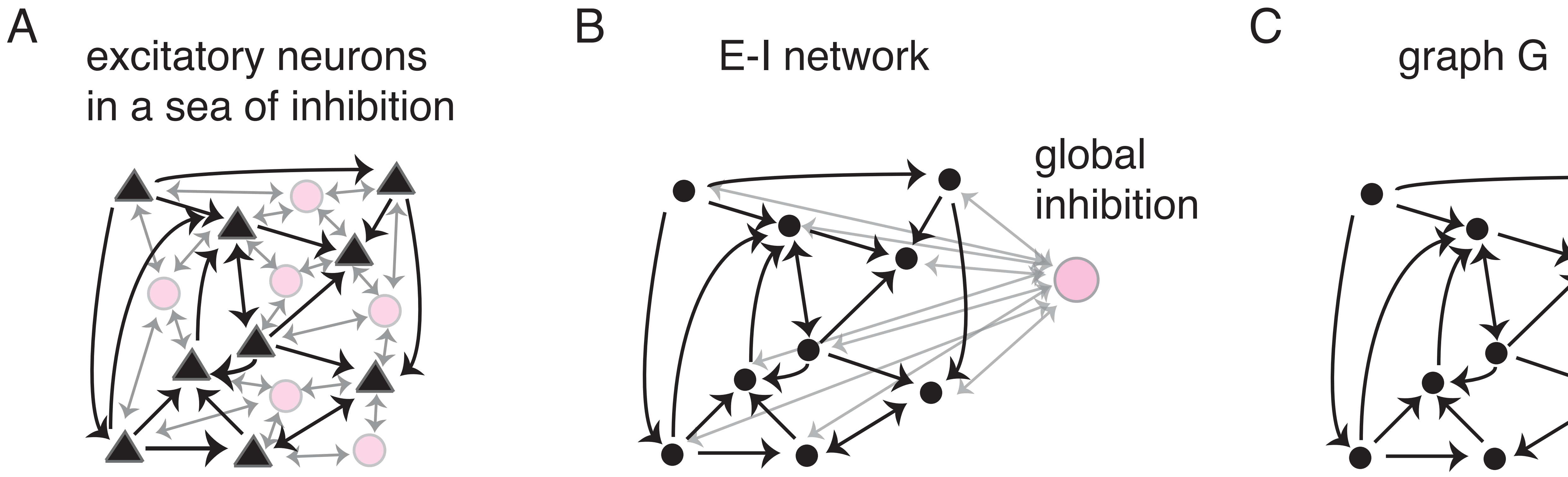
The gCTLN is defined by a graph G and two vectors of parameters:

$$W_{ij} = \begin{cases} -1+\varepsilon_j & \text{if } j \to i, \\ -1-\delta_j & \text{if } j \not\to i, \\ 0 & \text{if } i = j. \end{cases}$$

weak inhibition

strong inhibition

E-I TLNs from graphs



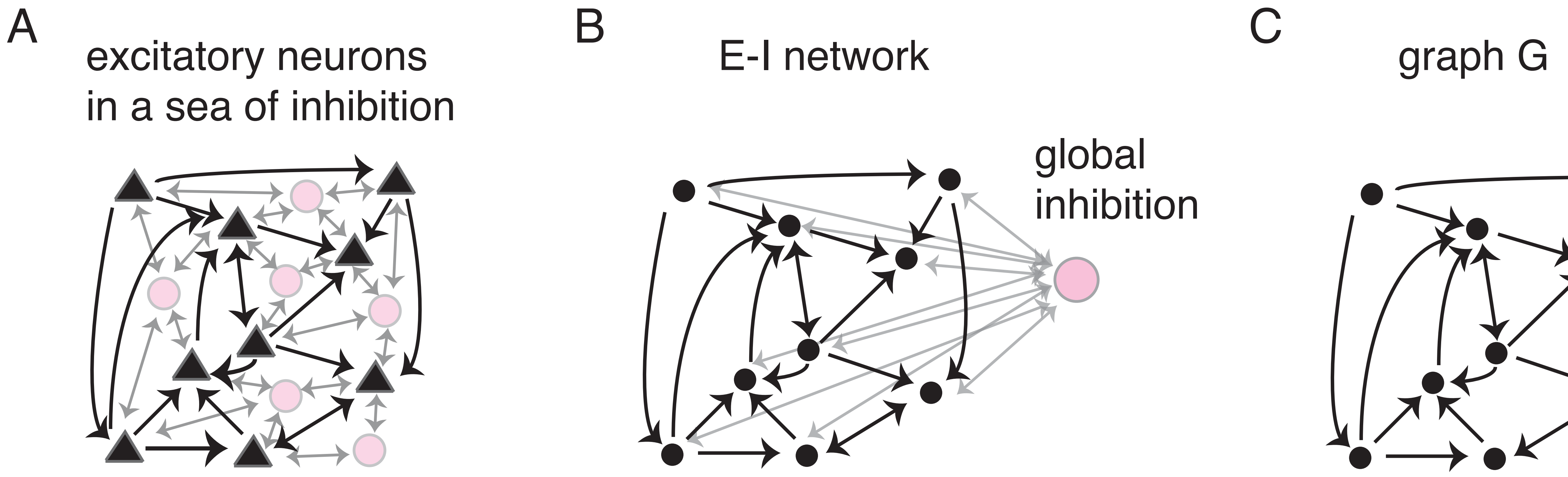
$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + W_{iI}(x_I - W_{Ii}x_i) + b_i\right]_+, \; i = 1, \ldots, n,$$

$$\frac{dx_I}{dt} = \frac{1}{\tau_I}\left(-x_I + \left[\sum_{j=1}^{n} W_{Ij}x_j + b_I\right]_+\right).$$

$$W_{ij} = \begin{cases} a_j & \text{if } j \to i \text{ in } G, \\ 0 & \text{if } j \not\to i \text{ in } G, \\ 0 & \text{if } i = j, \end{cases} \quad \text{and} \quad \begin{array}{rcl} W_{Ij} & = & c_j, \\ W_{iI} & = & -1, \\ W_{II} & = & 0. \end{array}$$

Curto 2025 (preprint soon!)

E-I TLNs from graphs



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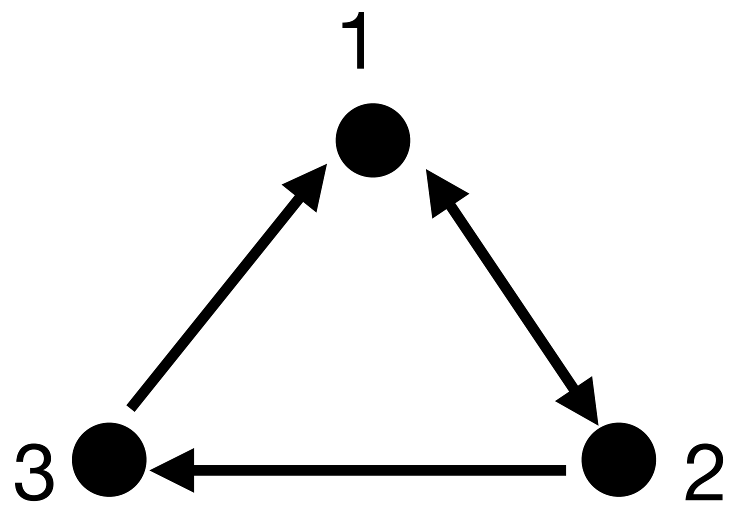
Example G:



W for E-I TLN

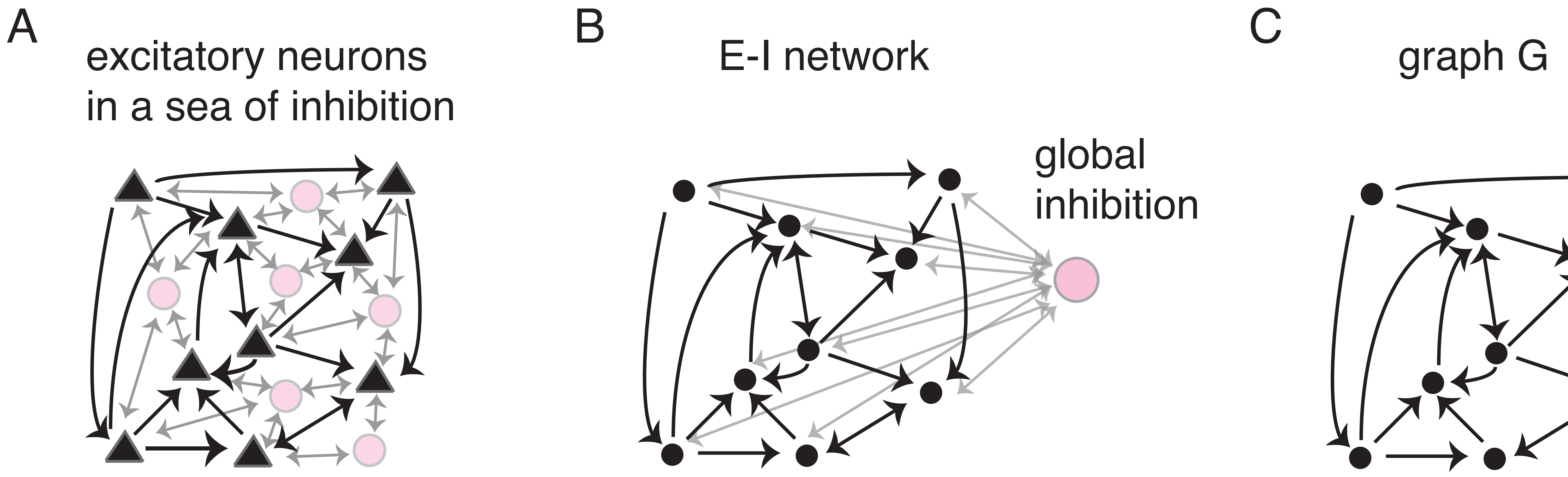
$$W = \begin{pmatrix} 0 & a_2 & a_3 & -1 \\ a_1 & 0 & 0 & -1 \\ 0 & a_2 & 0 & -1 \\ c_1 & c_2 & c_3 & 0 \end{pmatrix}$$

W for gCTLN

$$W = \begin{pmatrix} 0 & -1+\varepsilon_2 & -1+\varepsilon_3 \\ -1+\varepsilon_1 & 0 & -1-\delta_3 \\ -1-\delta_1 & -1+\varepsilon_2 & 0 \end{pmatrix}$$

Curto 2025 (preprint soon!)

There is a mapping from E-I TLNs to gCTLNs that preserves fixed points



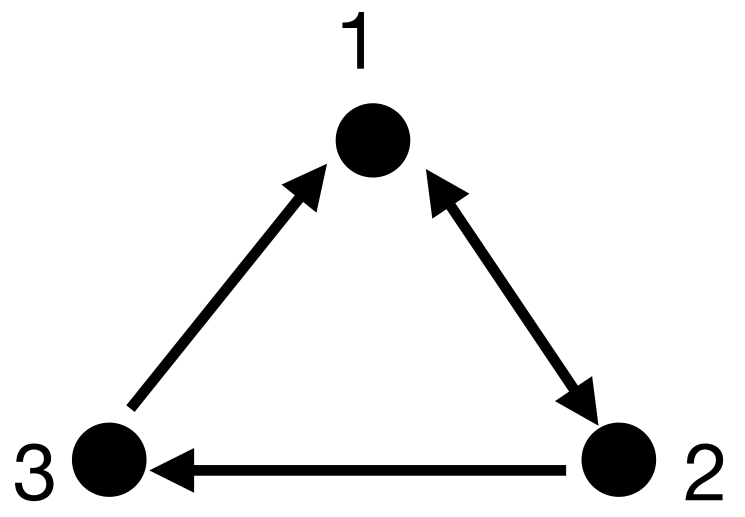
$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + \overset{\text{inhibitory interaction}}{\boxed{W_{iI}(x_I - W_{Ii}x_i)}} + b_i\right]_+ , \; i = 1, \dots, n,$$

$$\frac{dx_I}{dt} = \frac{1}{\tau_I}\left(-x_I + \left[\sum_{j=1}^{n} W_{Ij}x_j + b_I\right]_+\right).$$

Parameter mapping
to get the same
fixed points:

$$\begin{aligned} \varepsilon_j &= 1 + a_j - c_j, \\ \delta_j &= c_j - 1. \end{aligned}$$

Example G:



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see also: C. Lienkaemper, G. Ocker. Dynamics of clustered spiking networks via the CTLN model (2025)
https://arxiv.org/abs/2506.06234

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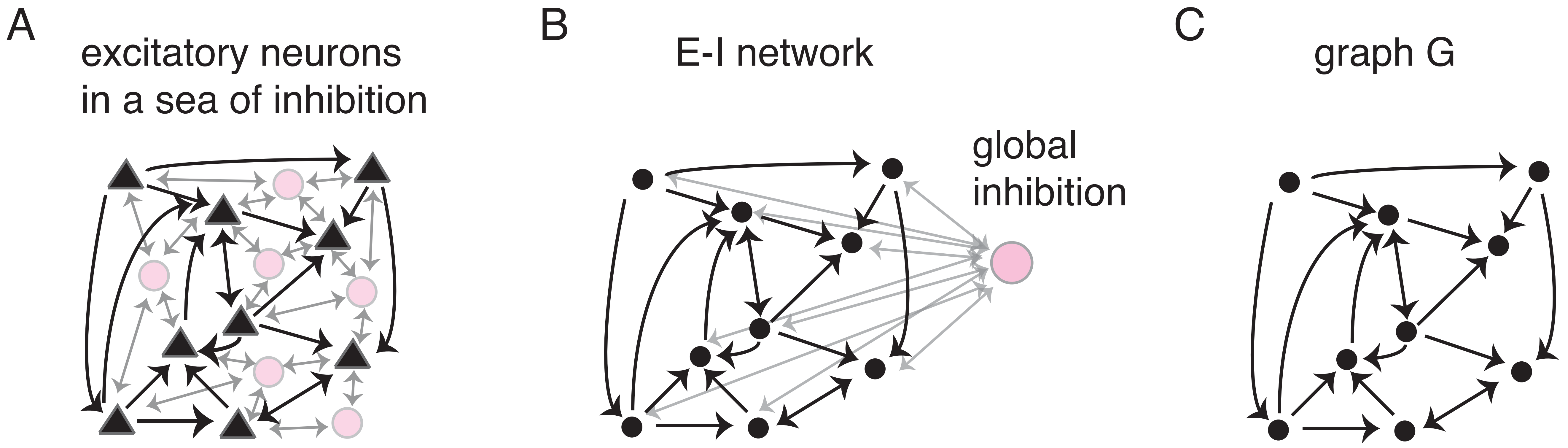
There is a mapping from E-I TLNs to gCTLNs that preserves fixed points



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Parameter mapping
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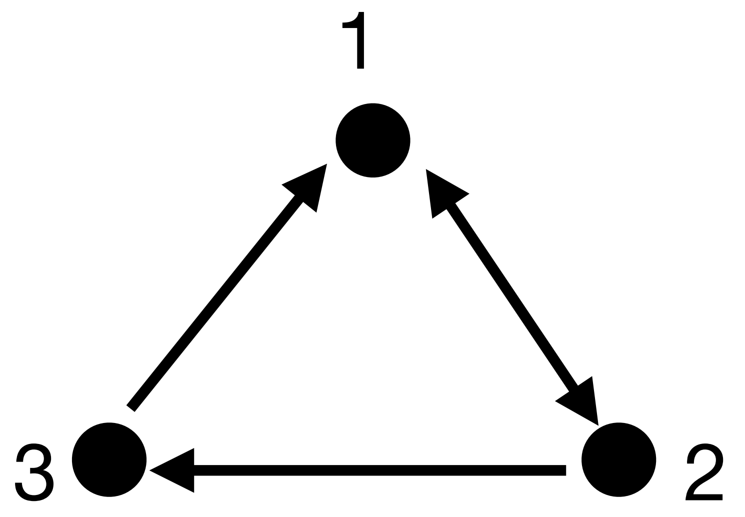
The mapping says nothing about the timescale of inhibition!

Example G:



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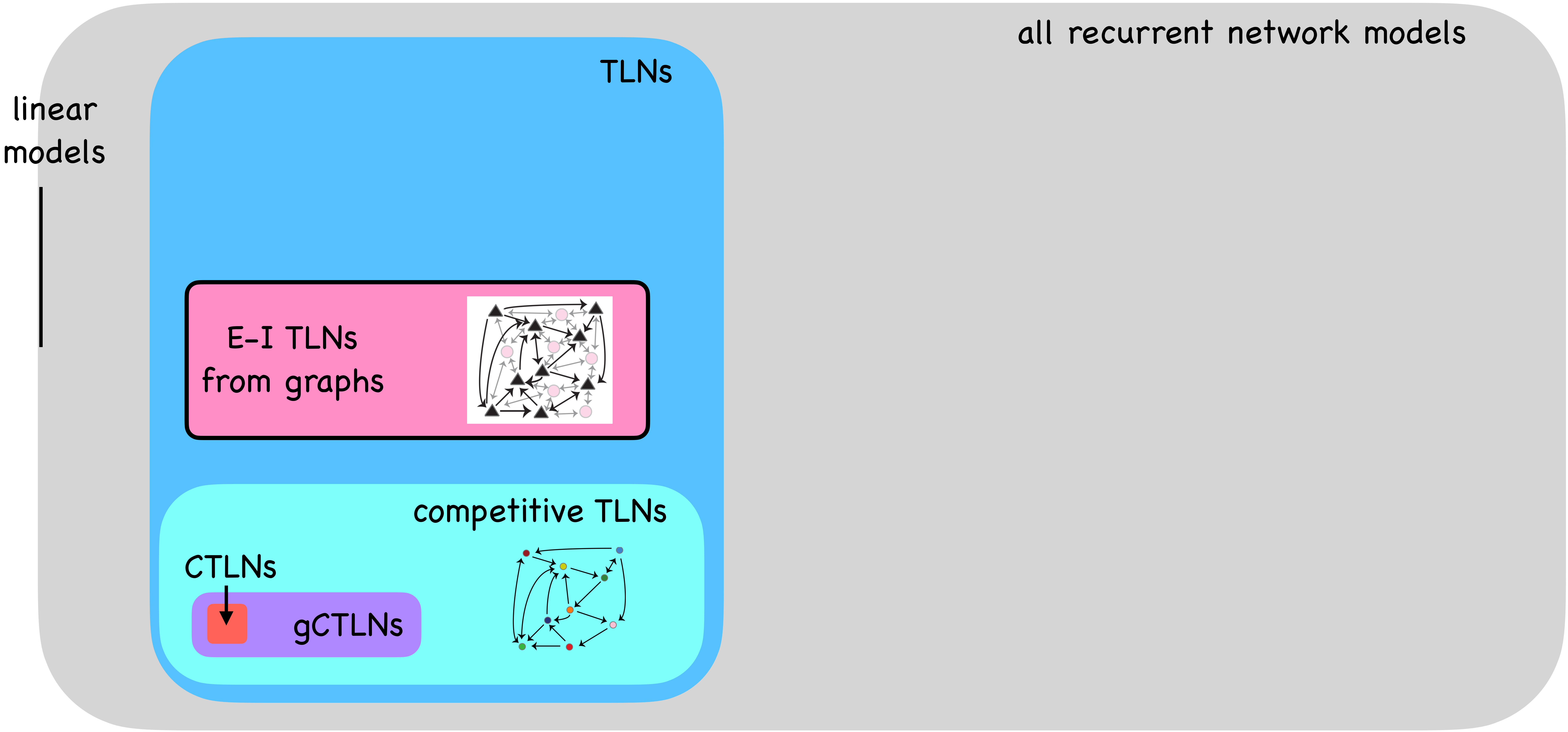
TLNs, CTLNs, and gCTLNs … and E-I TLNs from graphs



all recurrent network models

TLNs

linear models

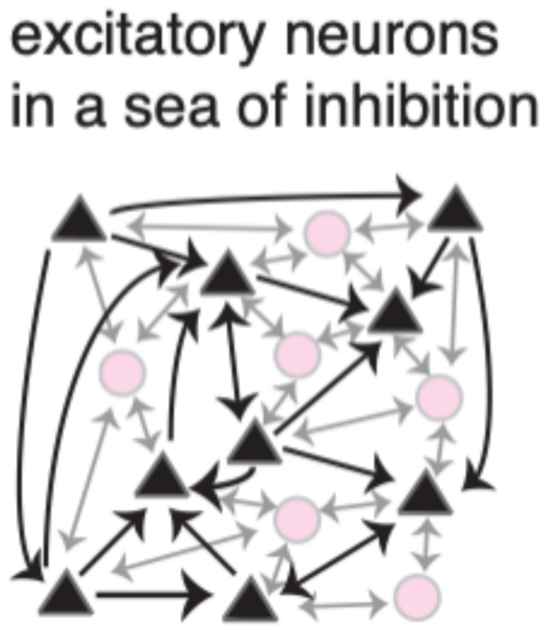
E-I TLNs from graphs

competitive TLNs

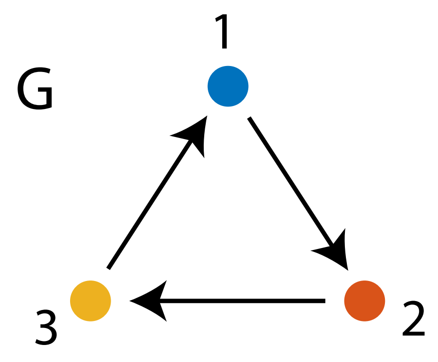
CTLNs

gCTLNs

Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?


excitatory neurons
in a sea of inhibition

A



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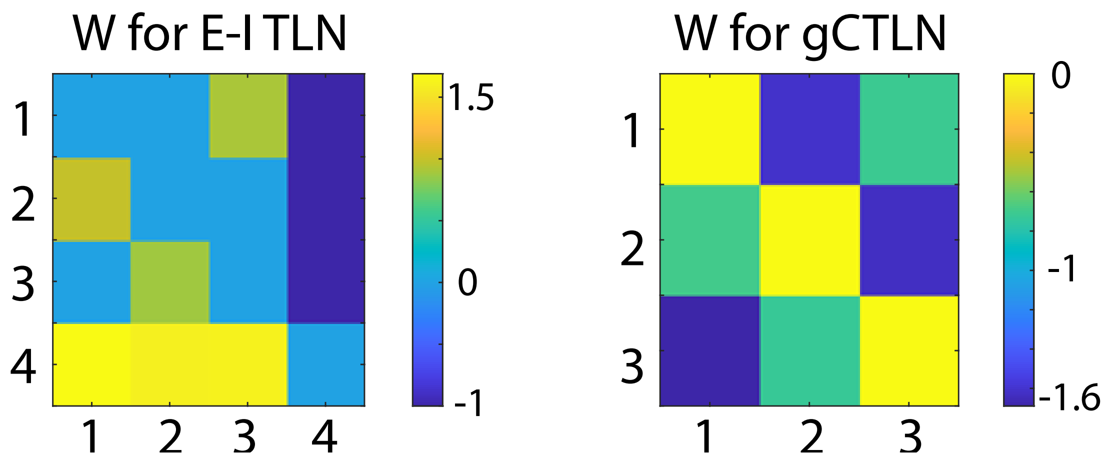
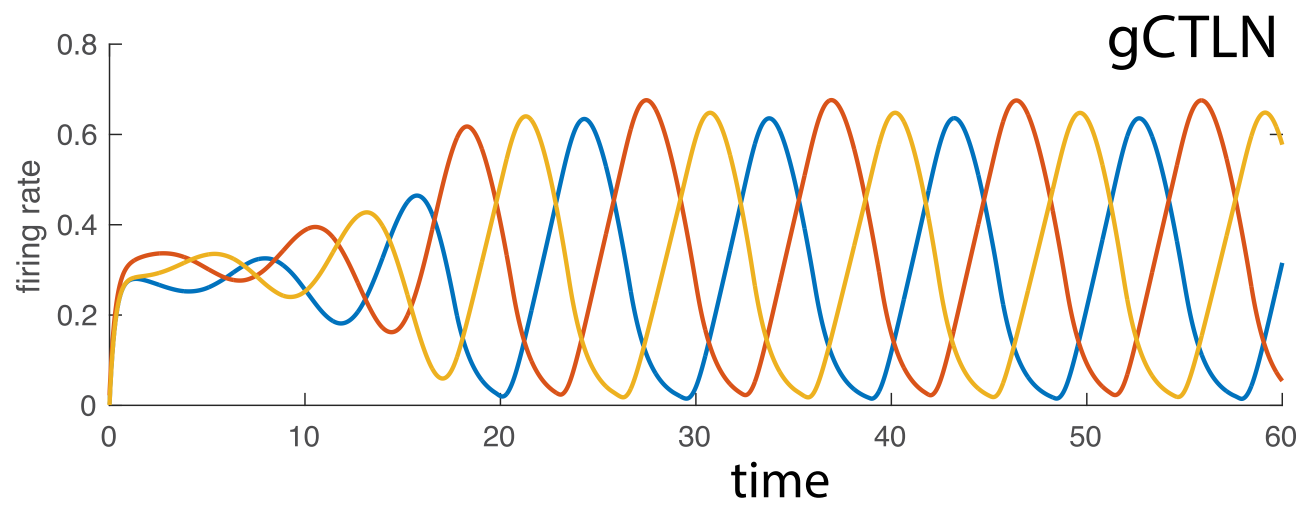
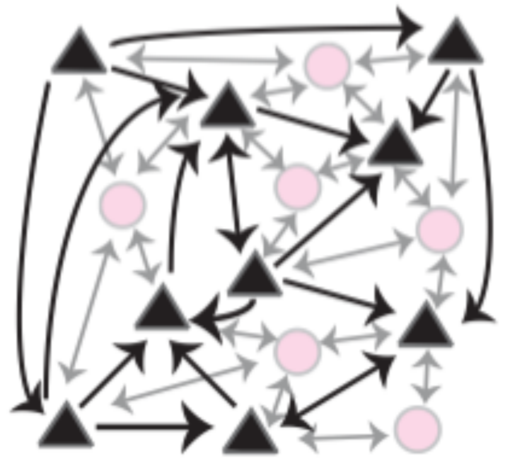
W for E-I TLN

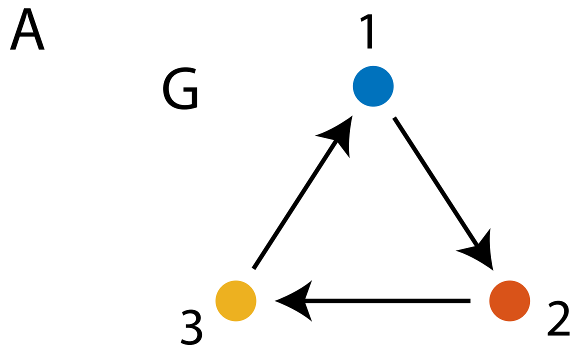




Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

excitatory neurons
in a sea of inhibition





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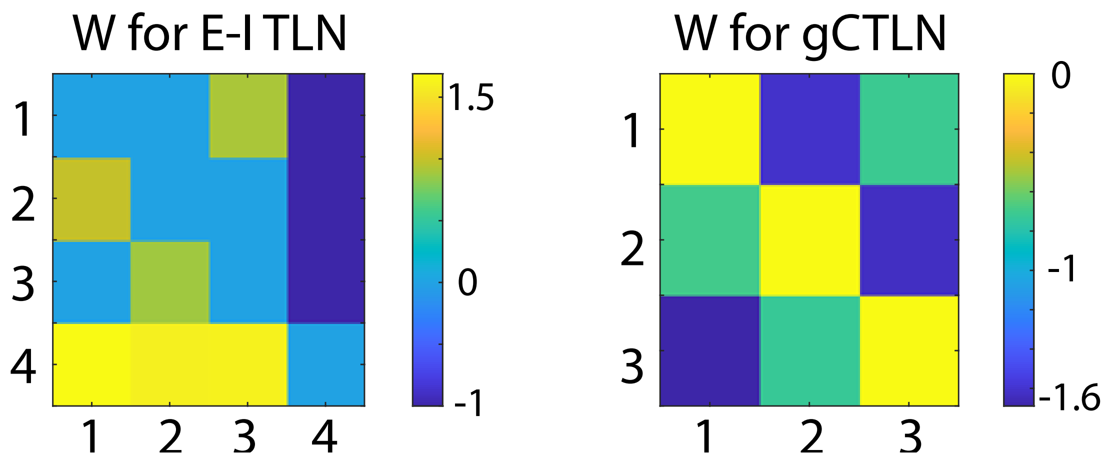
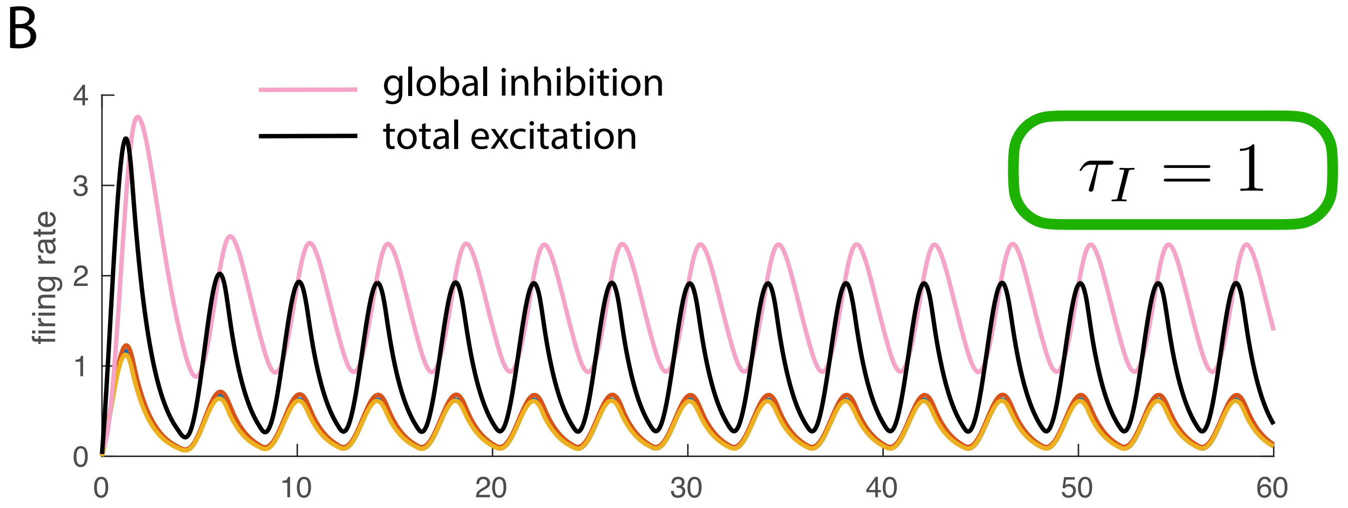
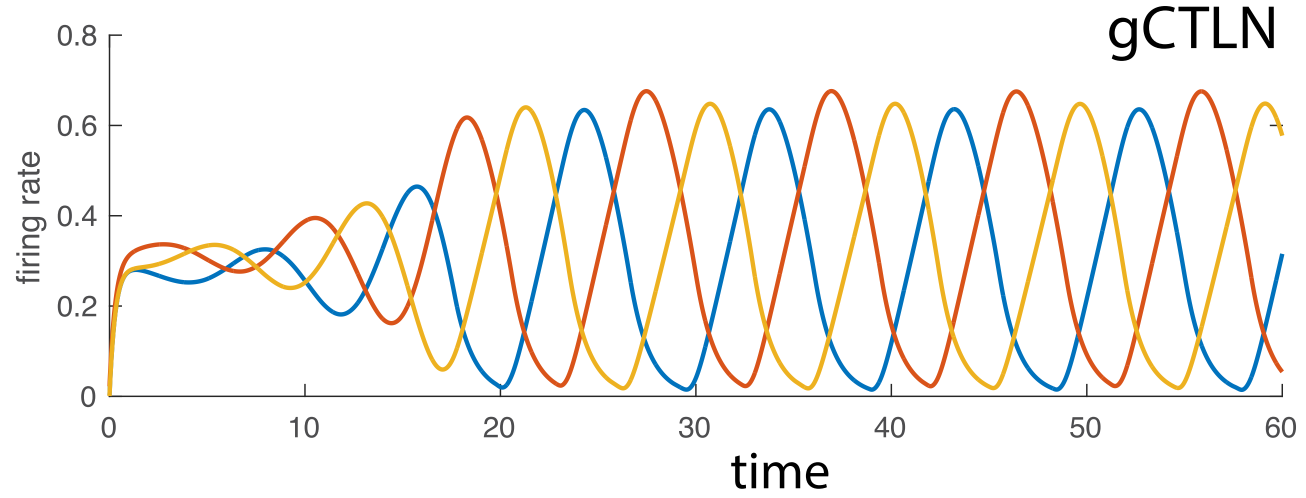
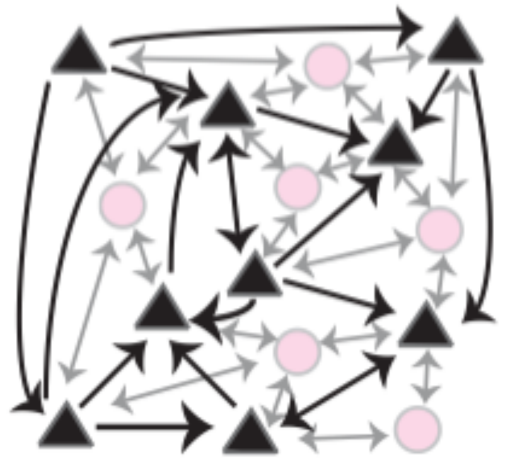




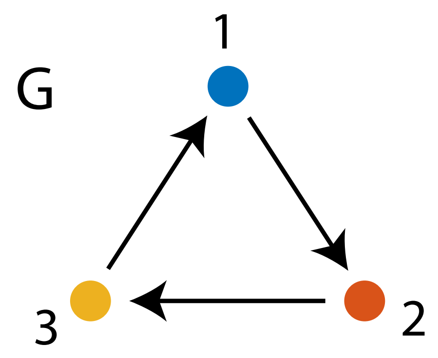


Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

excitatory neurons
in a sea of inhibition



A



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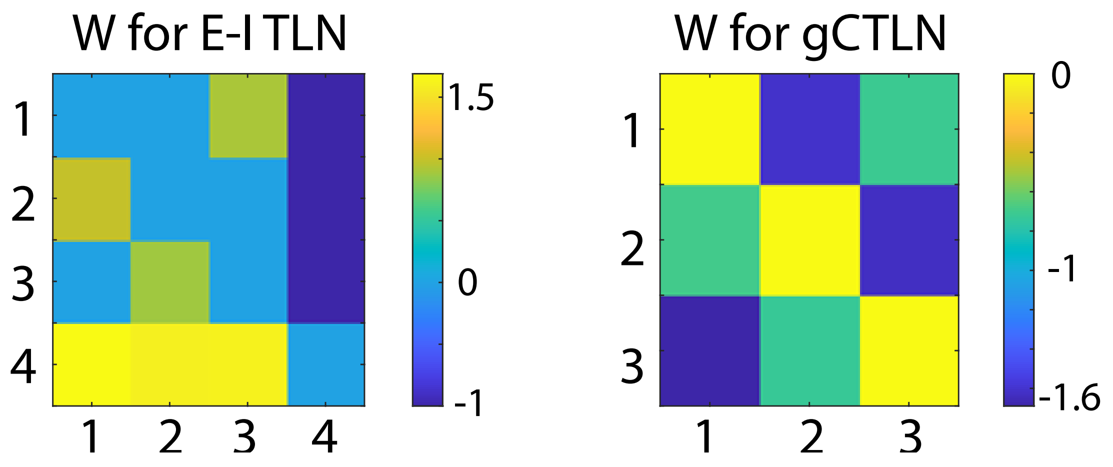
W for E-I TLN



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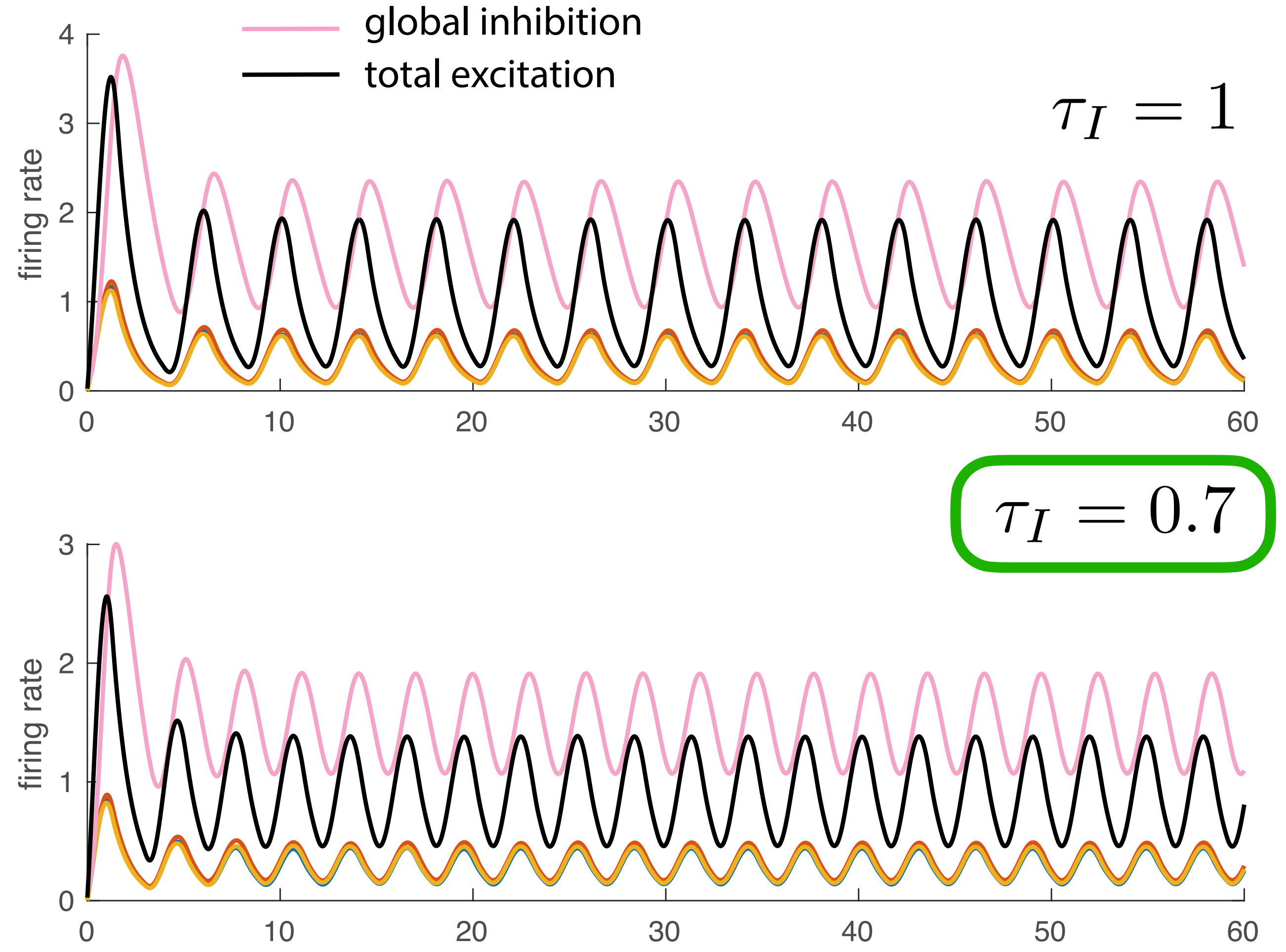

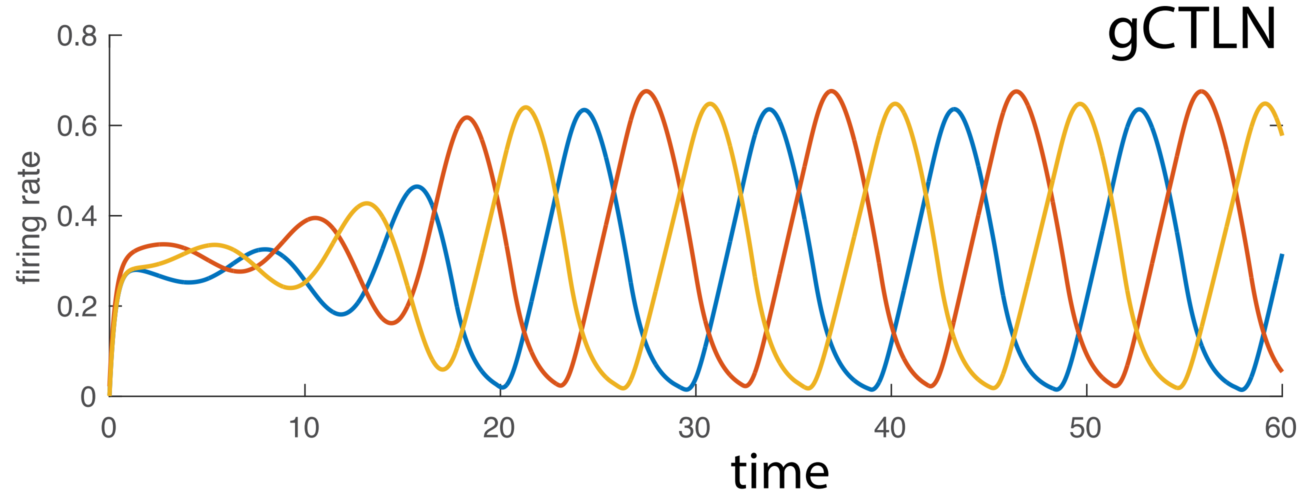

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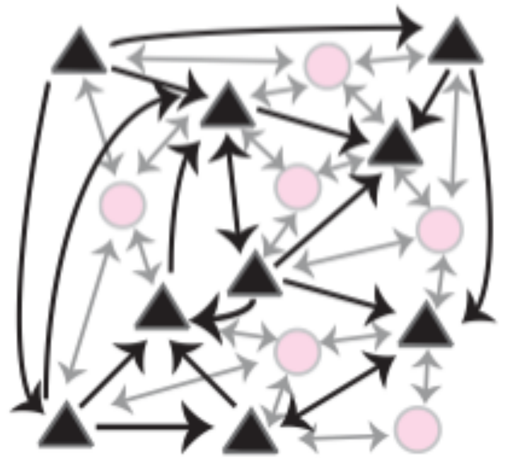

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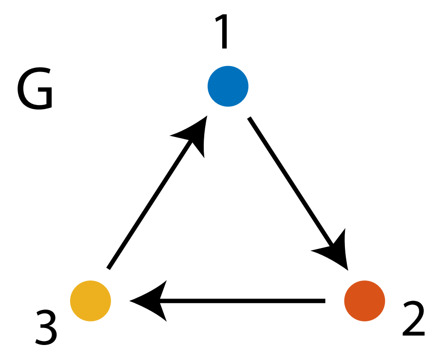

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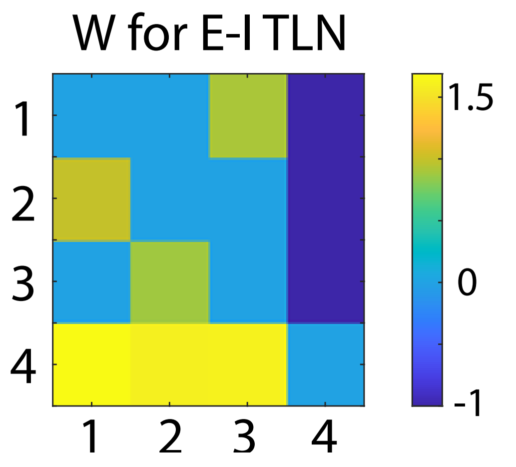

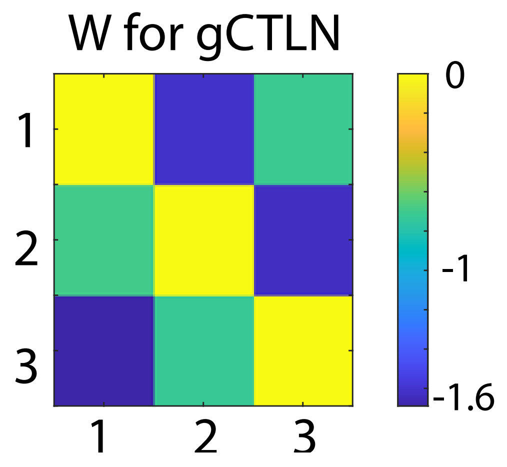

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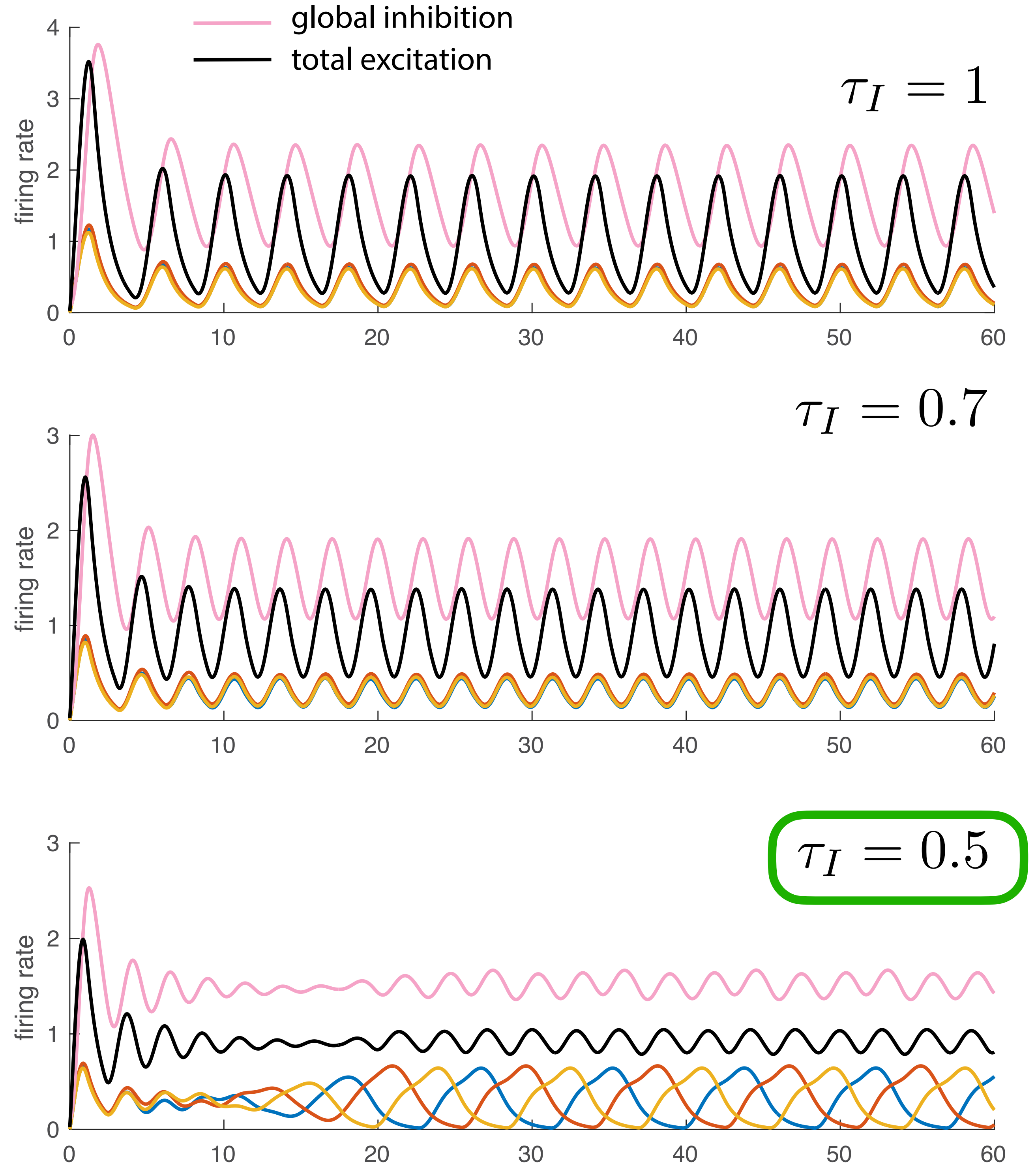

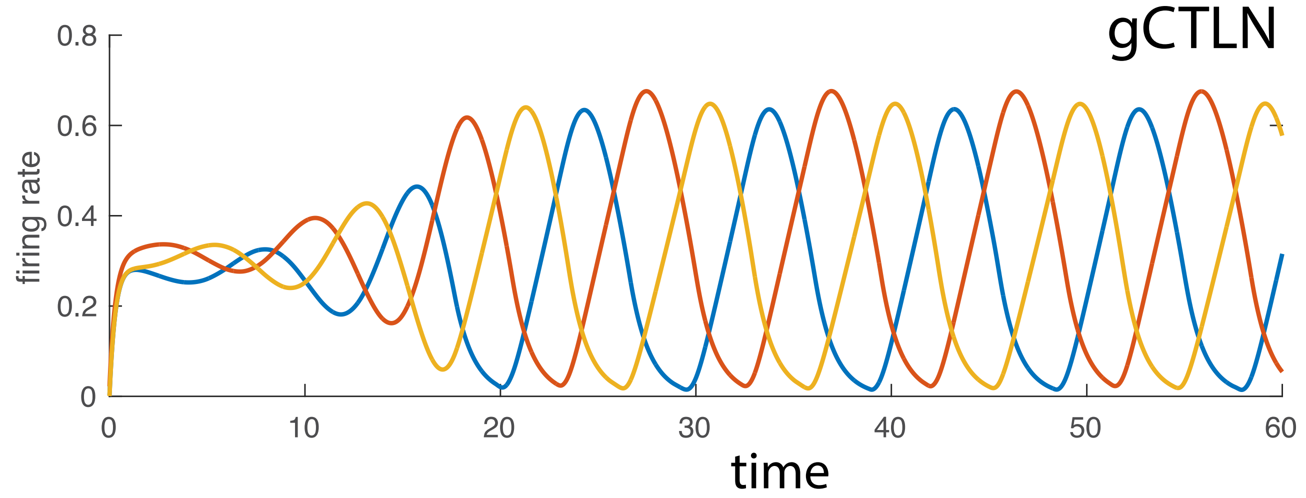

Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

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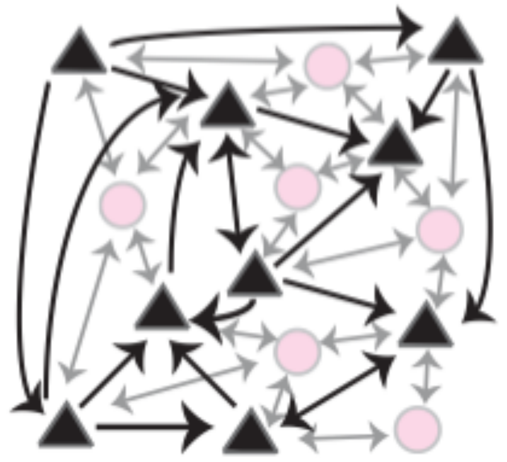
in a sea of inhibition



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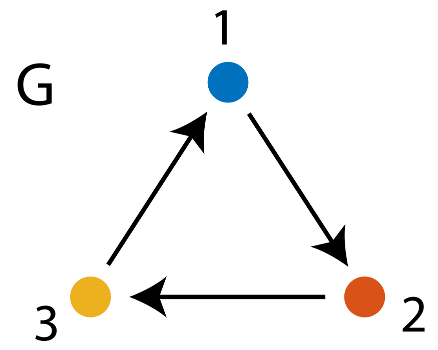

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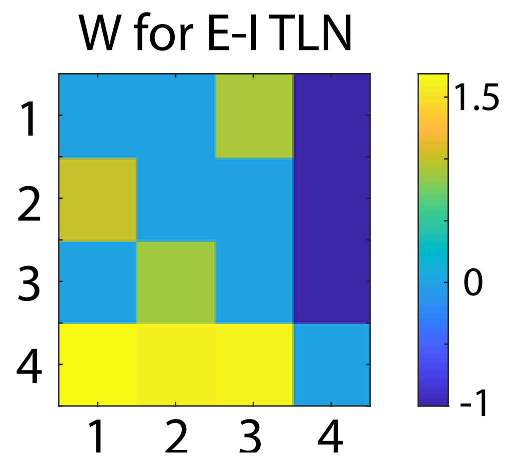
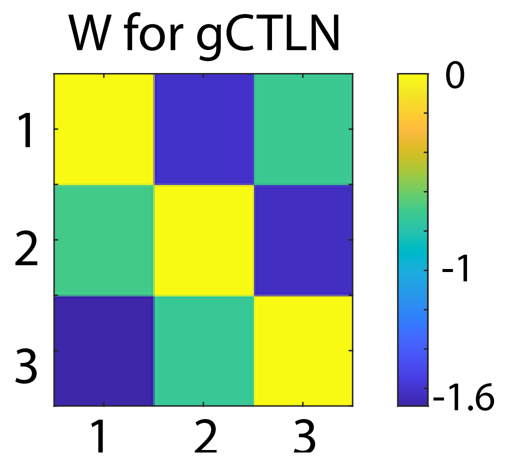
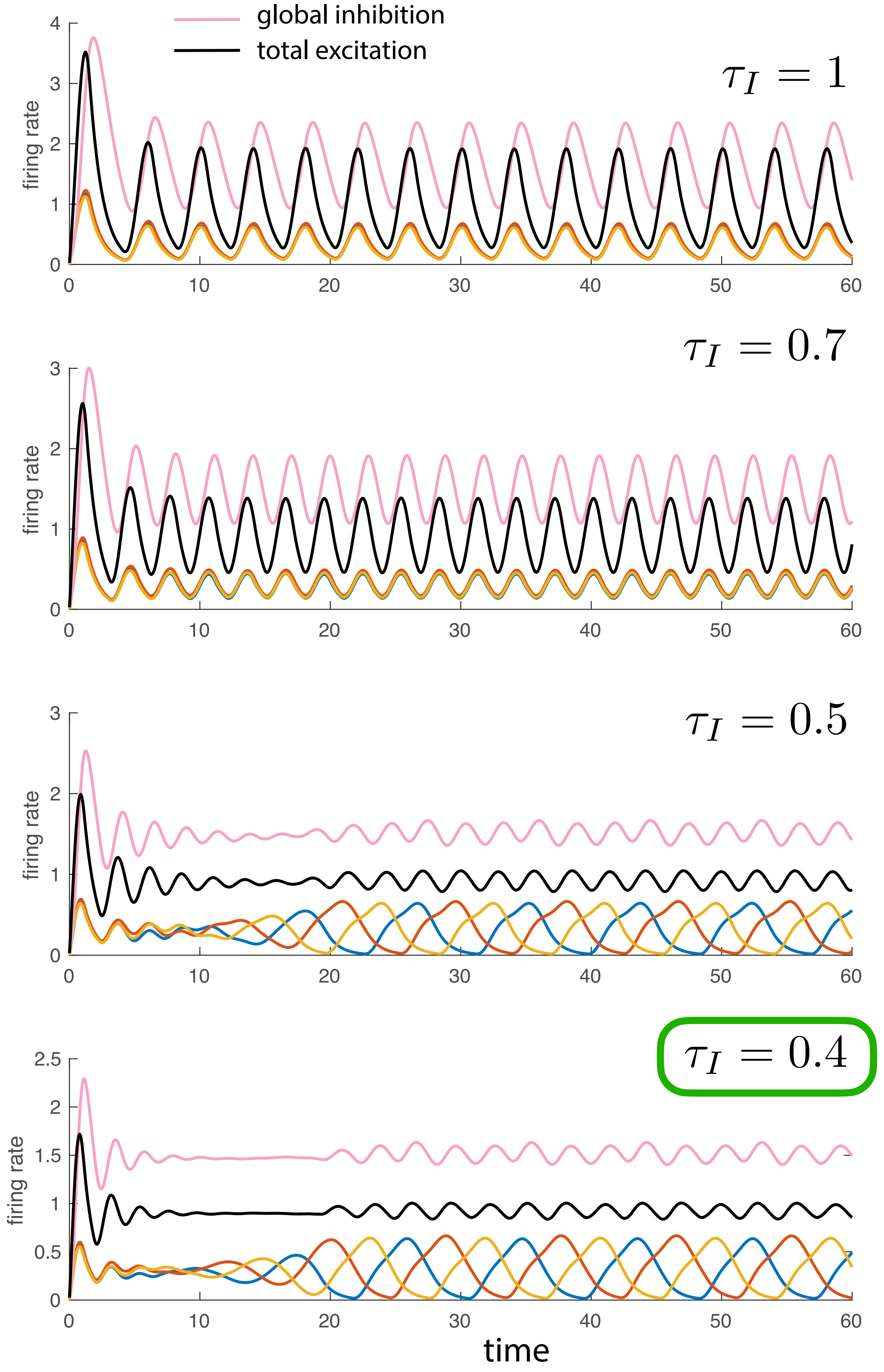
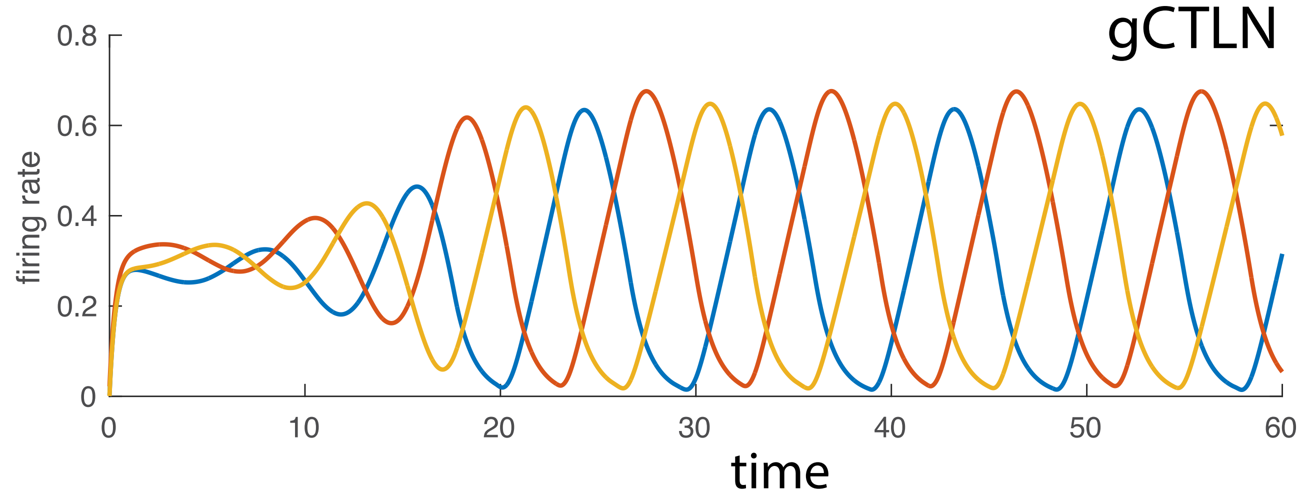




B





Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

excitatory neurons
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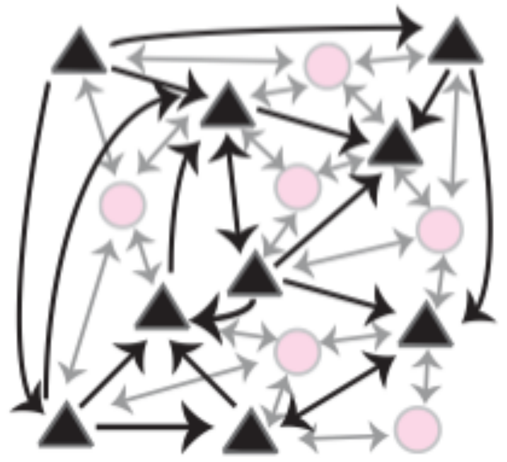

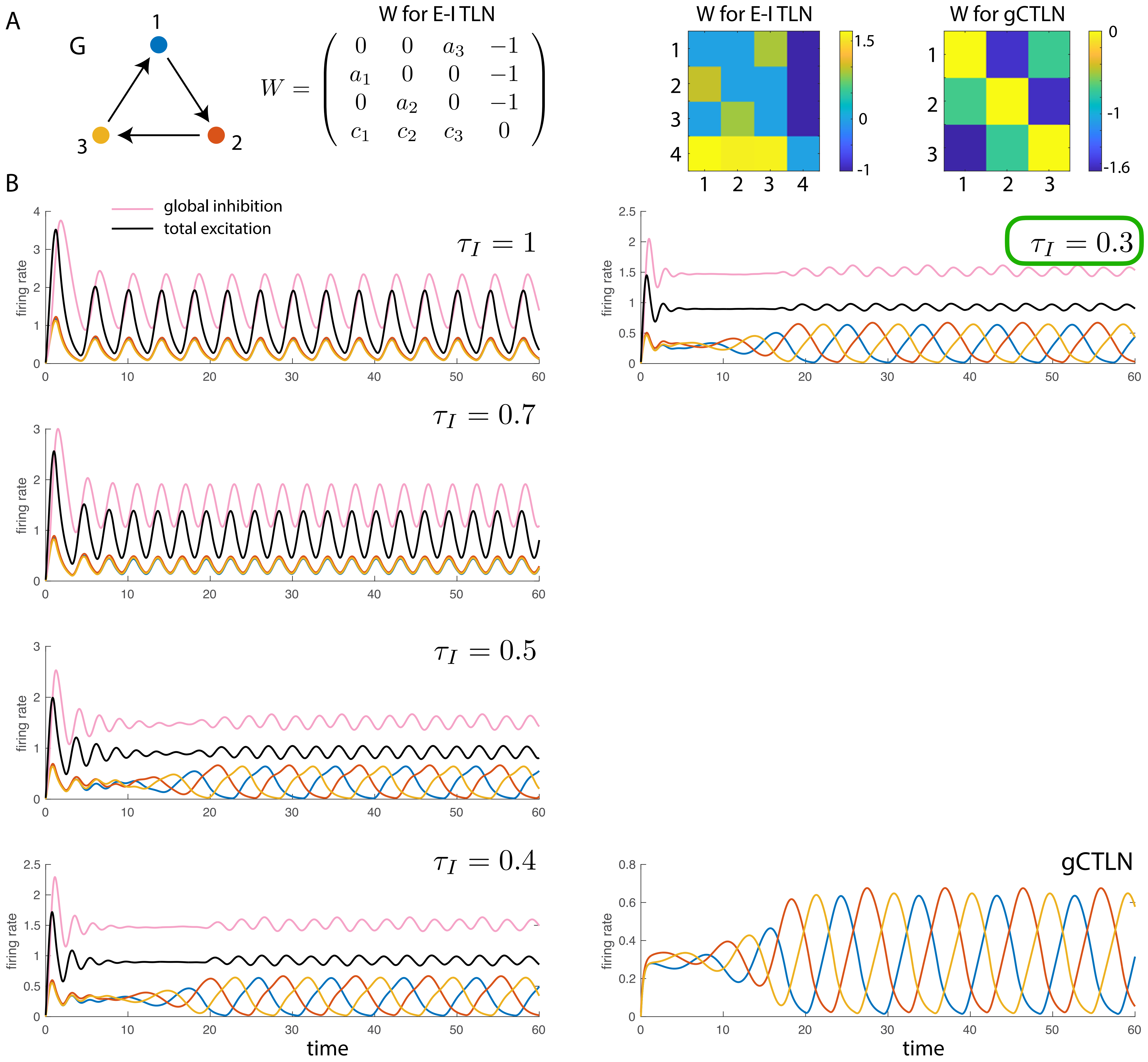

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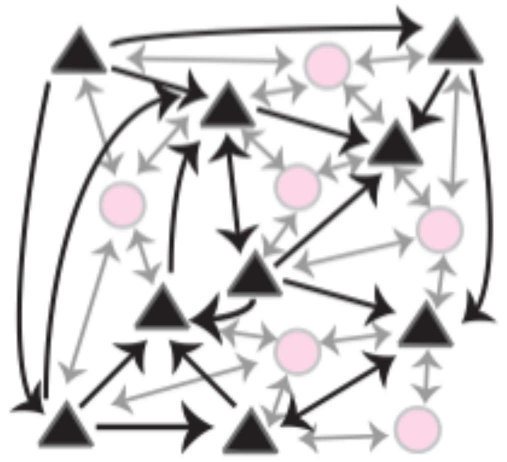
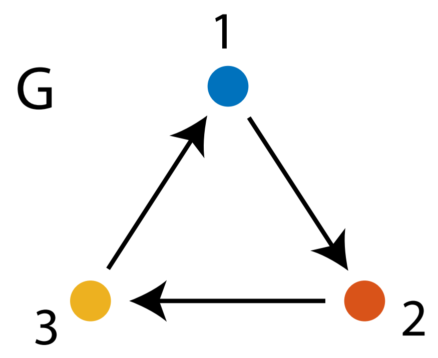
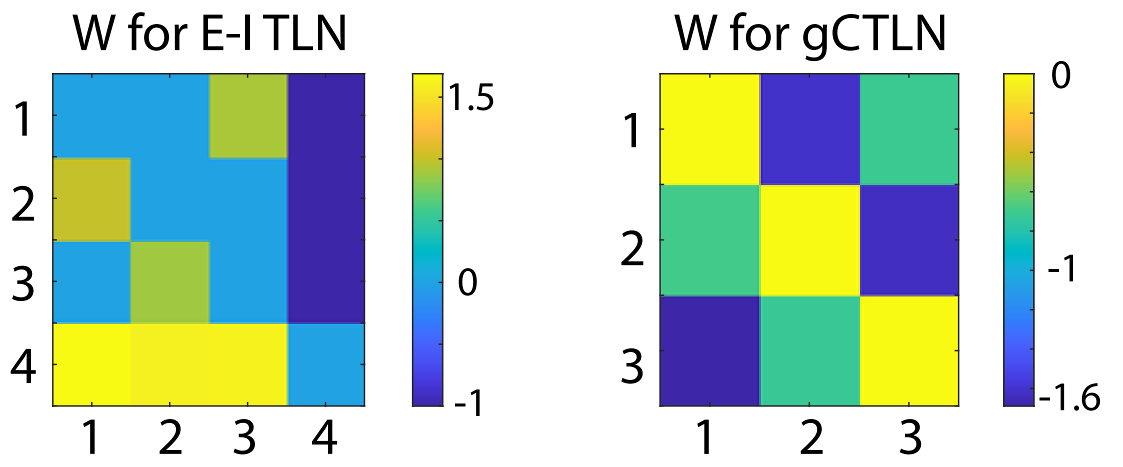


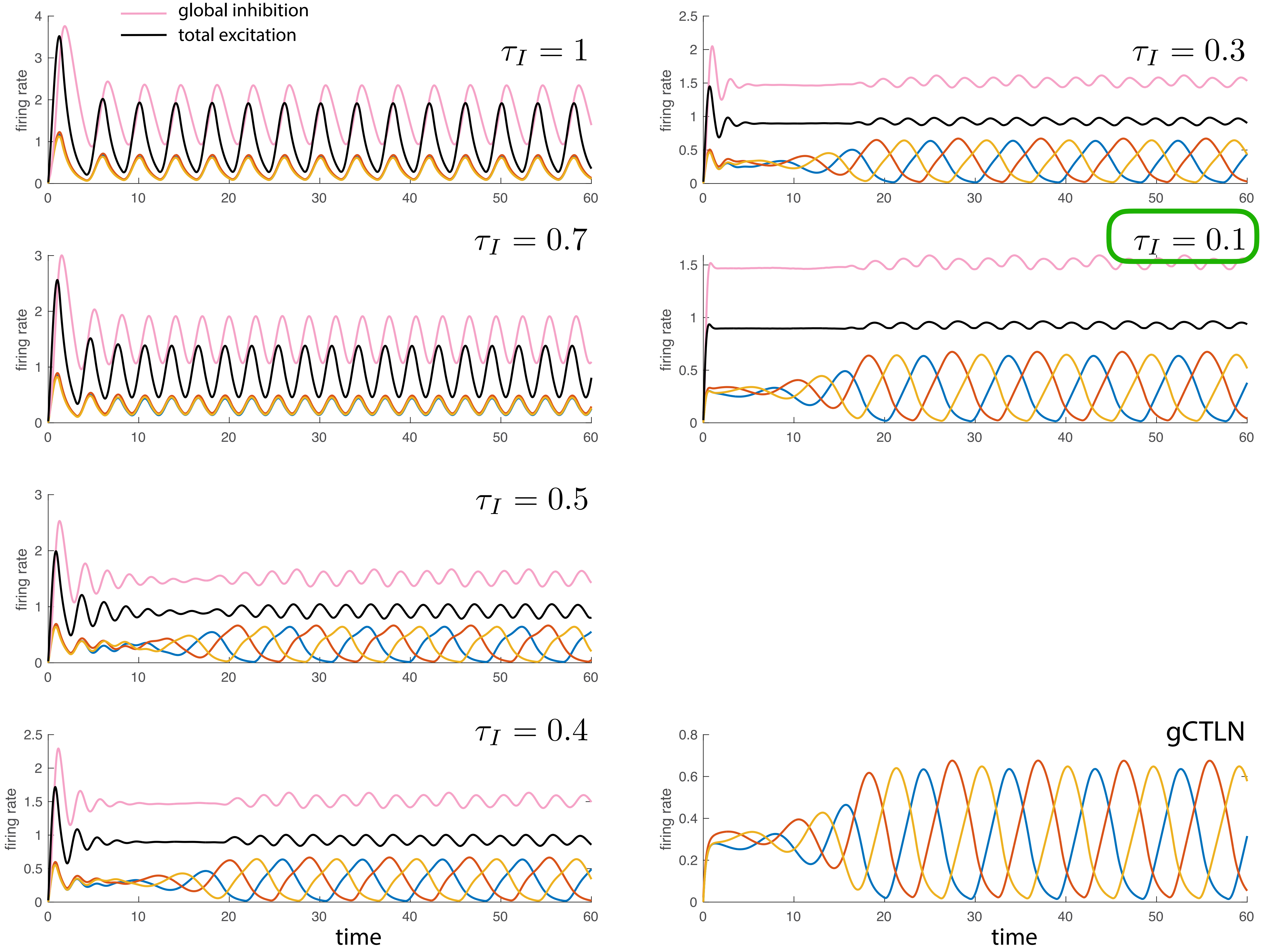
A



$$W = \begin{pmatrix} 0 & 0 & a_3 & -1 \\ a_1 & 0 & 0 & -1 \\ 0 & a_2 & 0 & -1 \\ c_1 & c_2 & c_3 & 0 \end{pmatrix}$$



B



Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

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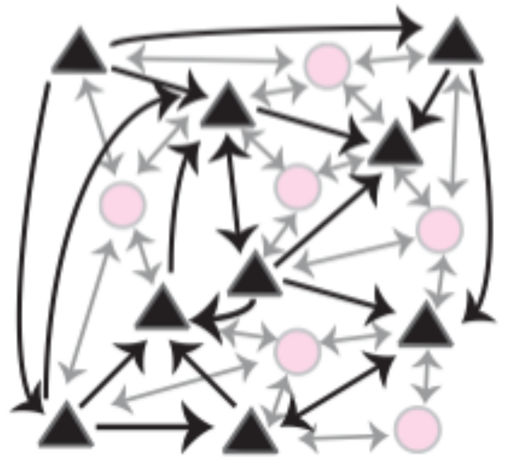
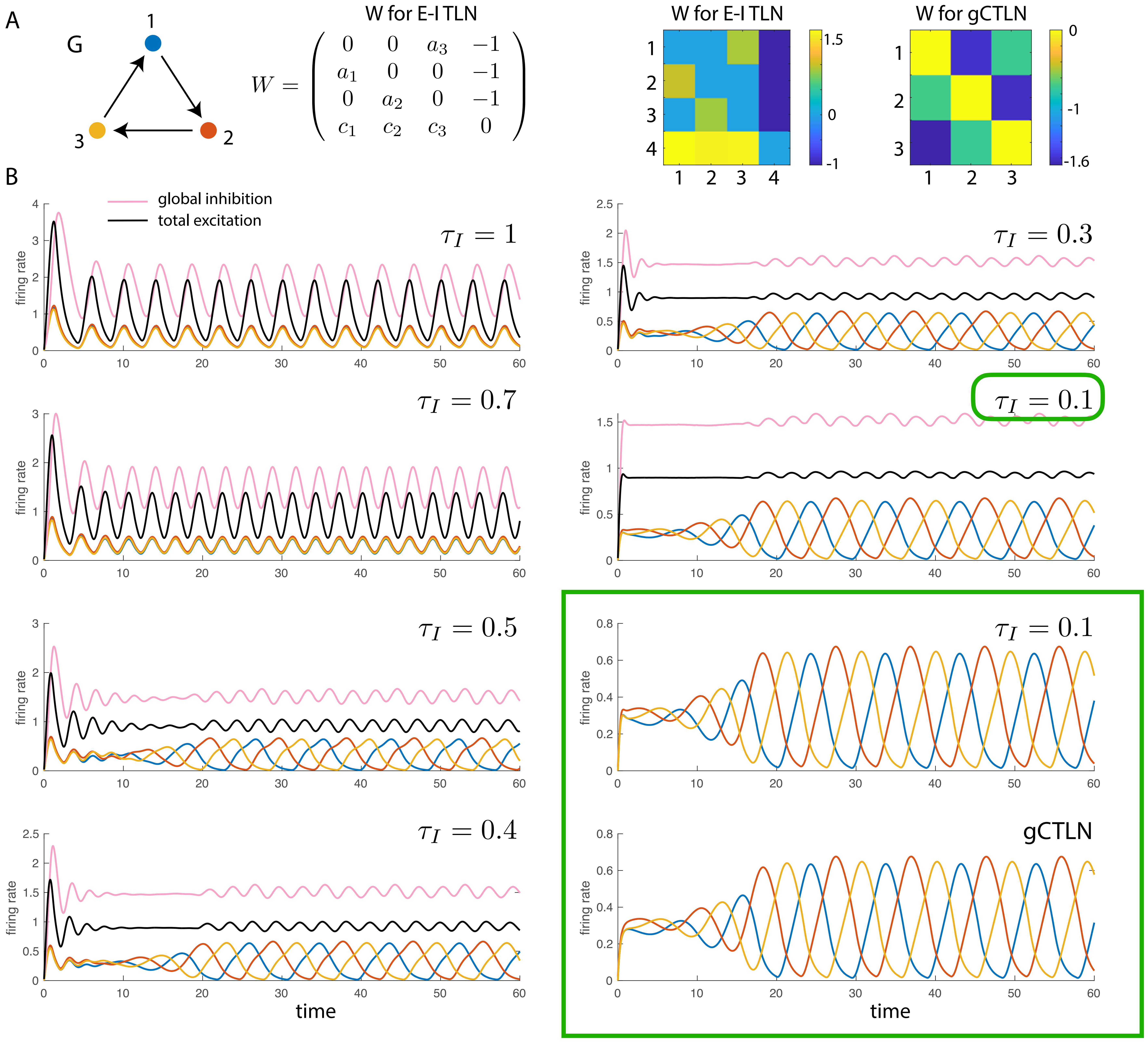
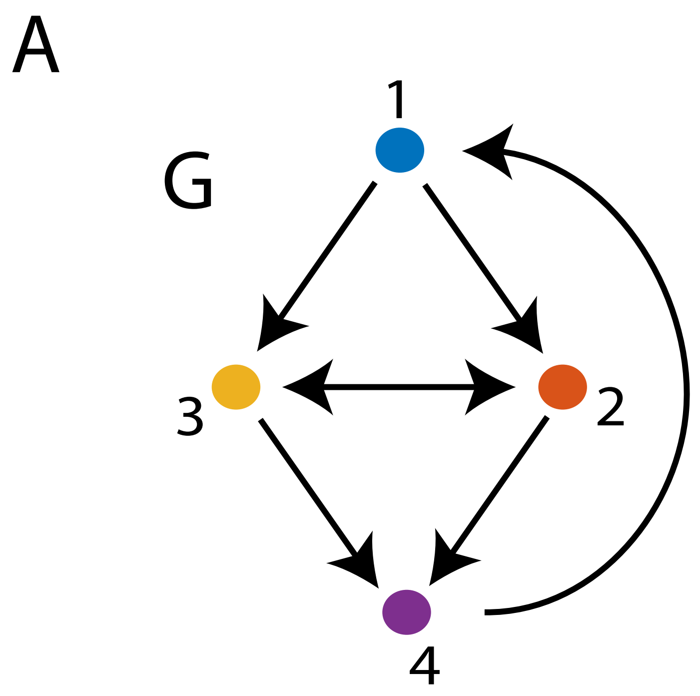




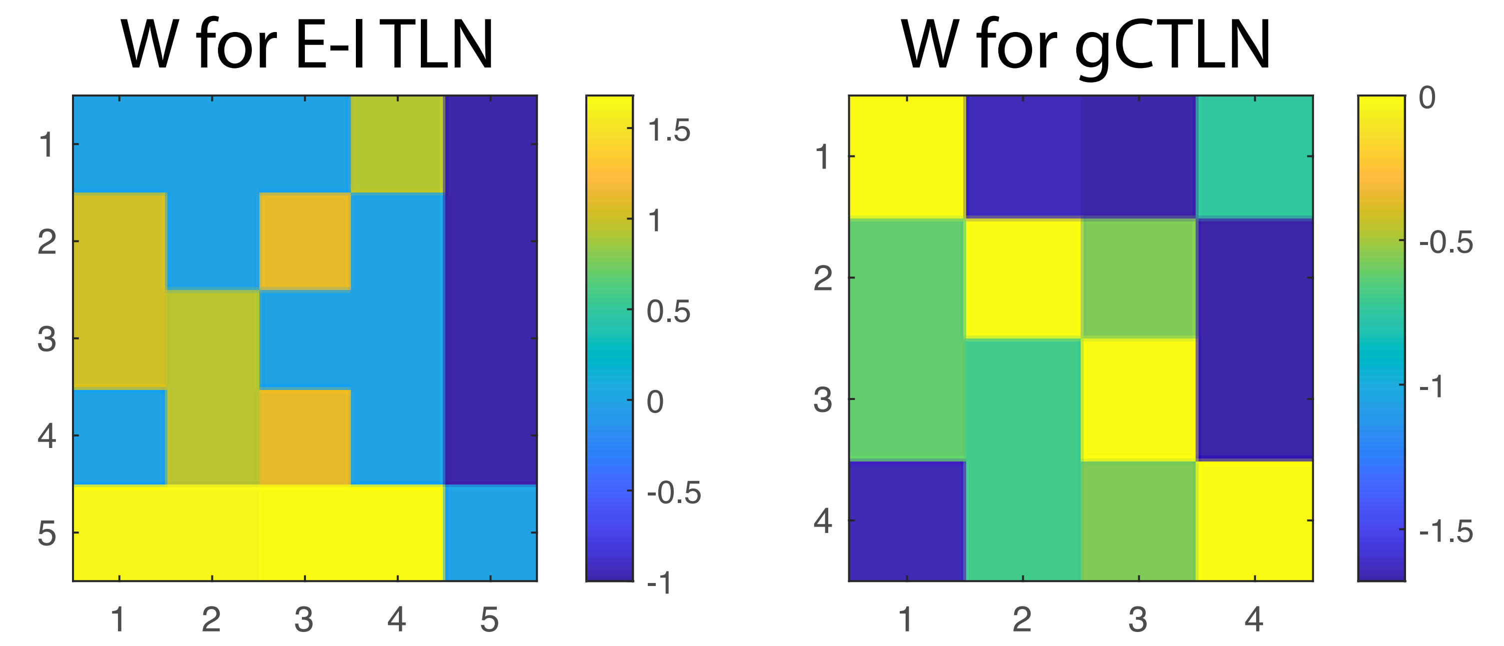
$$W = \begin{pmatrix} 0 & 0 & a_3 & -1 \\ a_1 & 0 & 0 & -1 \\ 0 & a_2 & 0 & -1 \\ c_1 & c_2 & c_3 & 0 \end{pmatrix}$$

Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

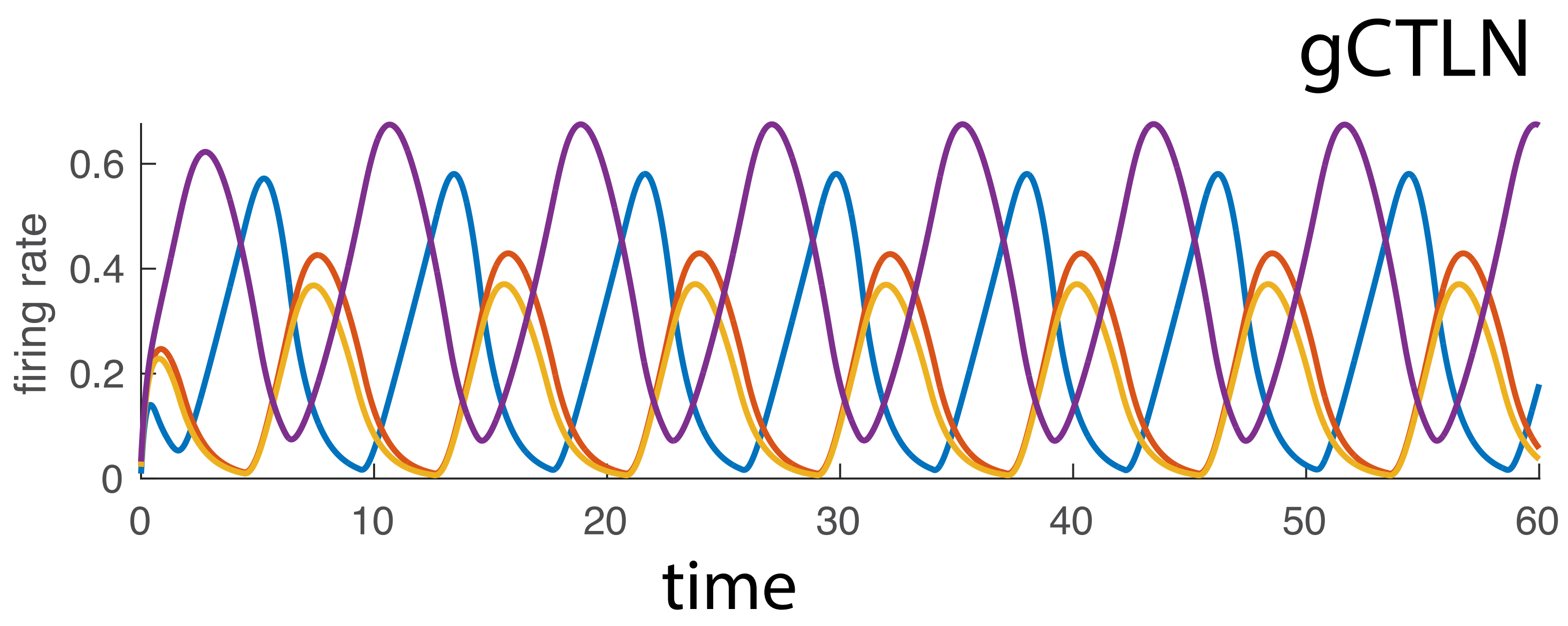


W for E-I TLN

$$W = \begin{pmatrix} 0 & 0 & 0 & a_4 & -1 \\ a_1 & 0 & a_3 & 0 & -1 \\ a_1 & a_2 & 0 & 0 & -1 \\ 0 & a_2 & a_3 & 0 & -1 \\ c_1 & c_2 & c_3 & c_4 & 0 \end{pmatrix}$$

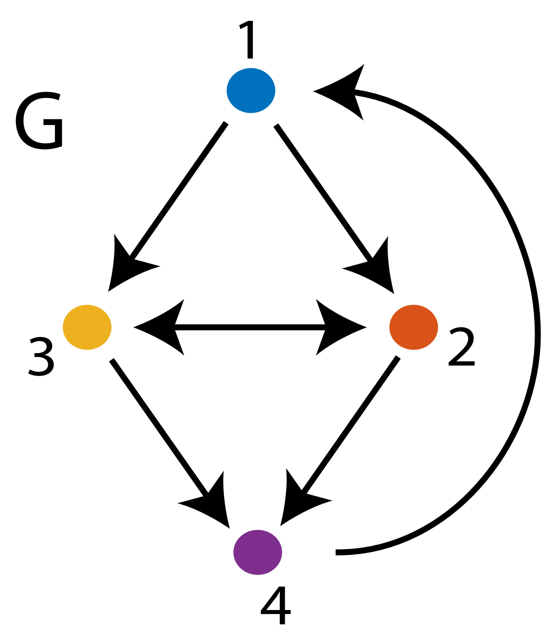

W for E-I TLN

W for gCTLN


gCTLN

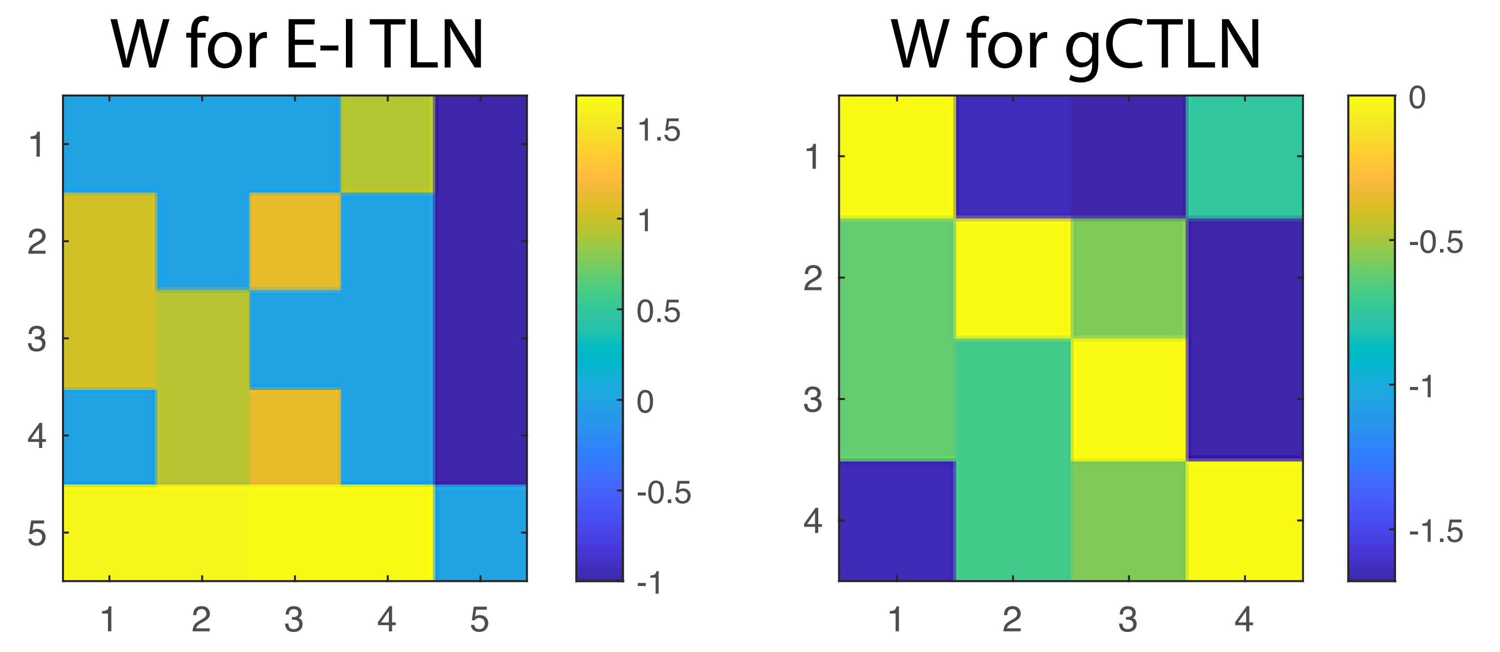
Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

A

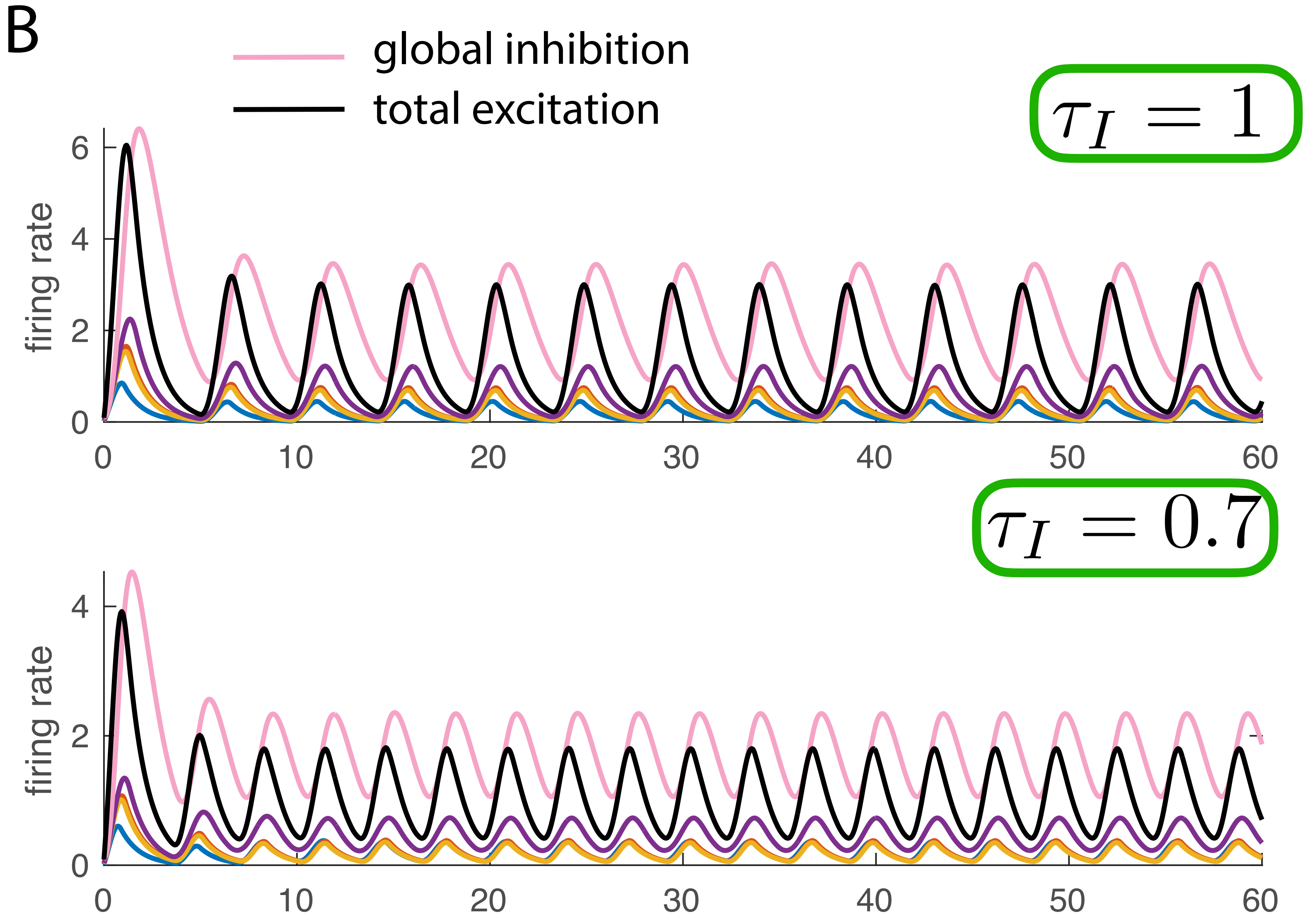


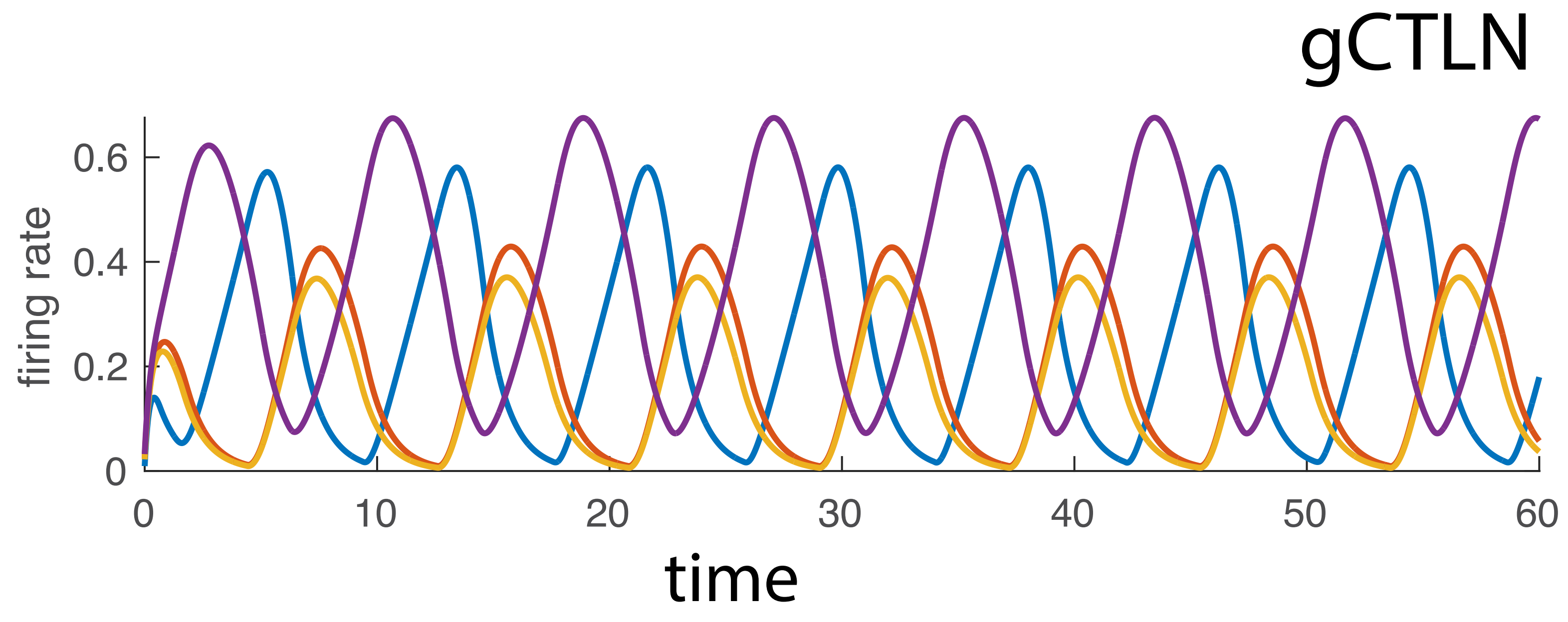
$$W = \begin{pmatrix} 0 & 0 & 0 & a_4 & -1 \\ a_1 & 0 & a_3 & 0 & -1 \\ a_1 & a_2 & 0 & 0 & -1 \\ 0 & a_2 & a_3 & 0 & -1 \\ c_1 & c_2 & c_3 & c_4 & 0 \end{pmatrix}$$

W for E-I TLN



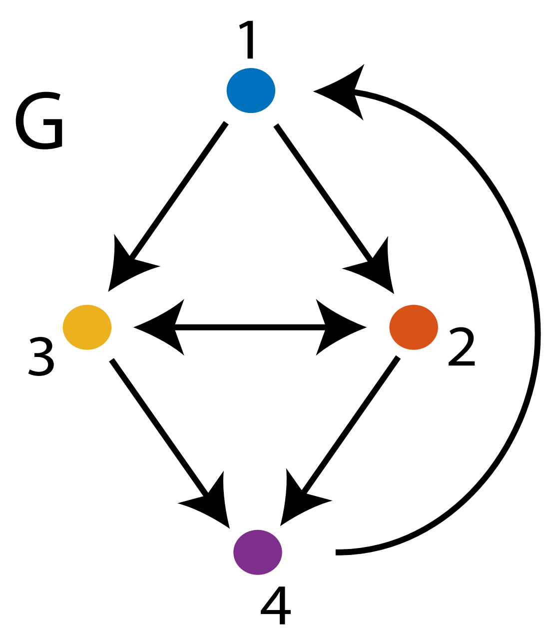
B





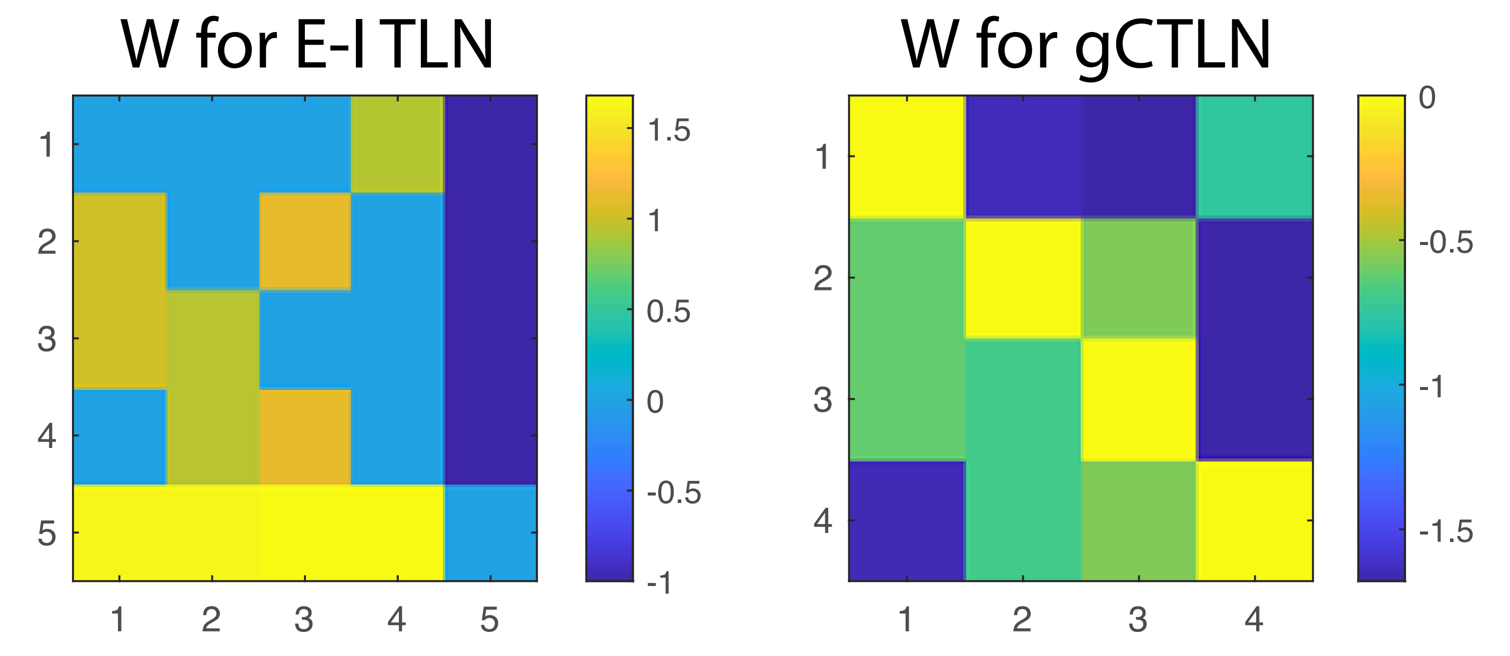
Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?

A

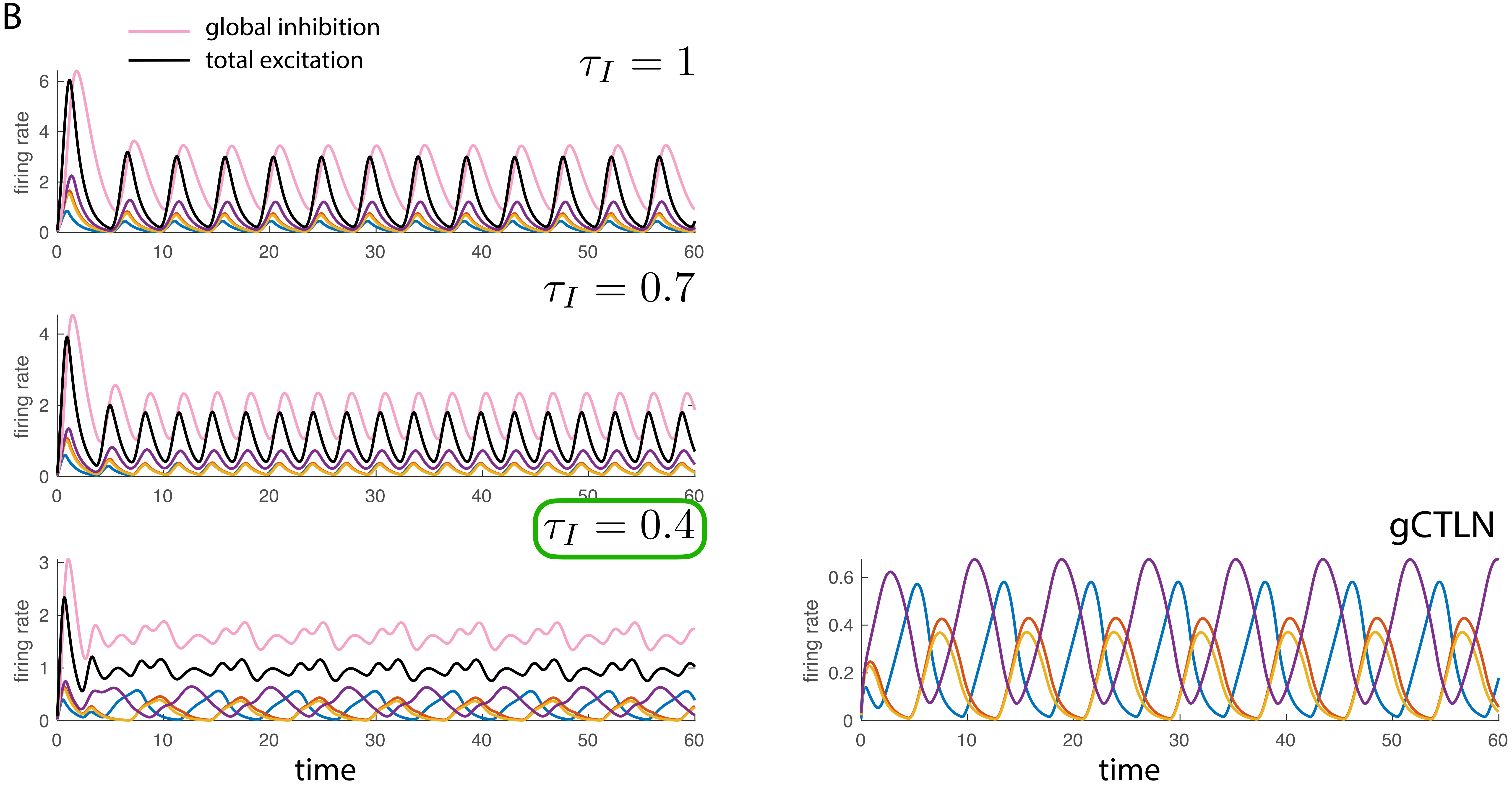


W for E-I TLN

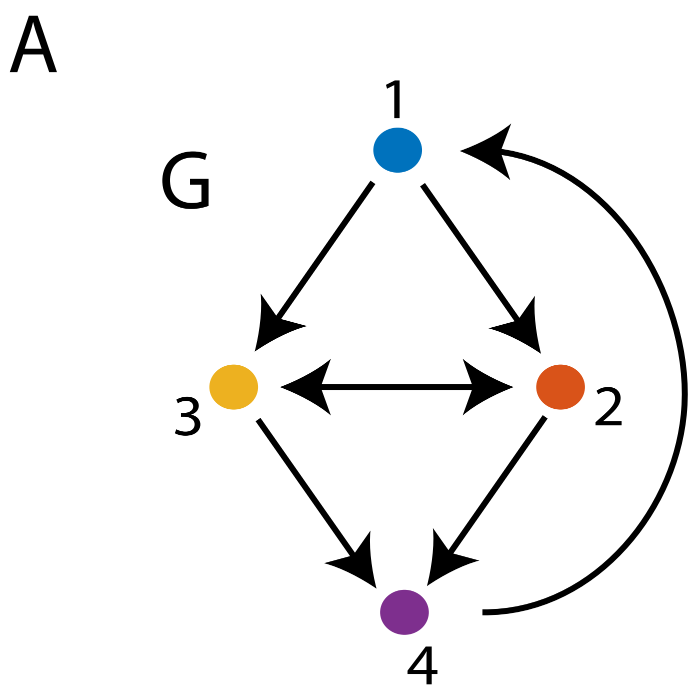
$$W = \begin{pmatrix} 0 & 0 & 0 & a_4 & -1 \\ a_1 & 0 & a_3 & 0 & -1 \\ a_1 & a_2 & 0 & 0 & -1 \\ 0 & a_2 & a_3 & 0 & -1 \\ c_1 & c_2 & c_3 & c_4 & 0 \end{pmatrix}$$



B

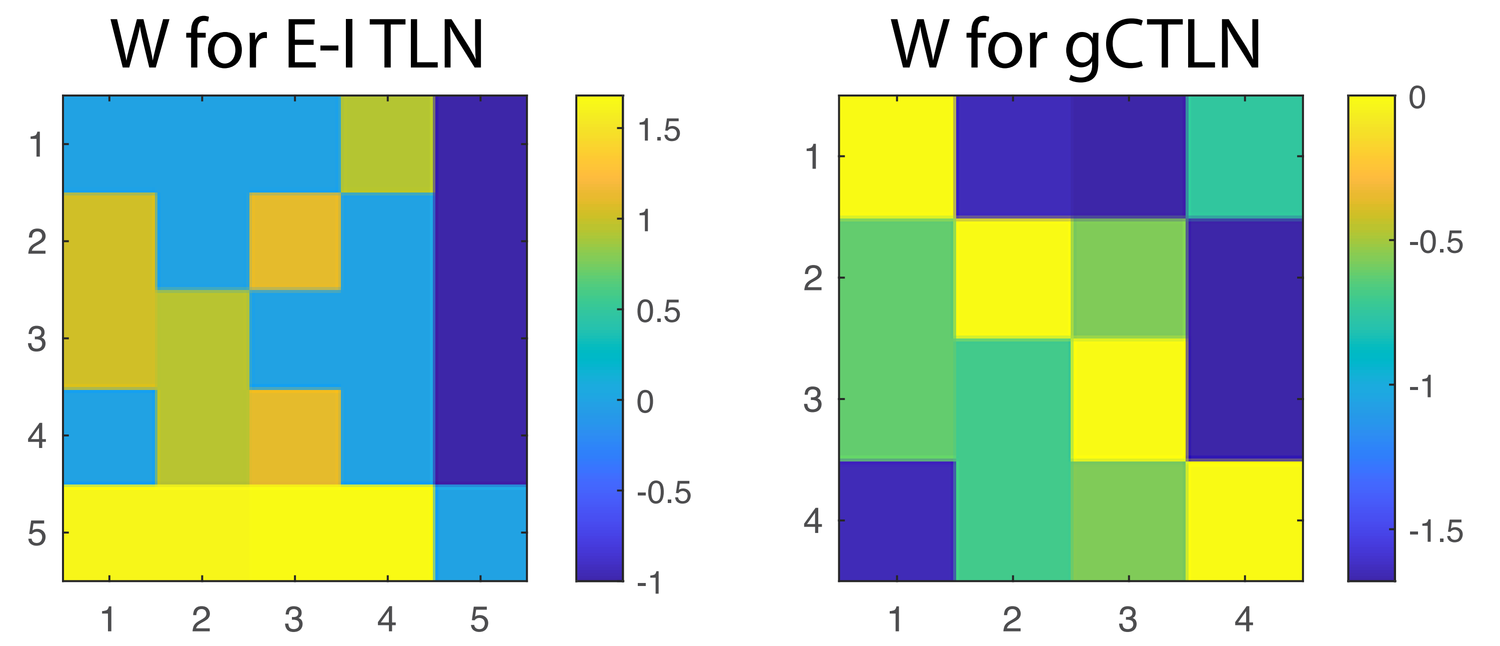


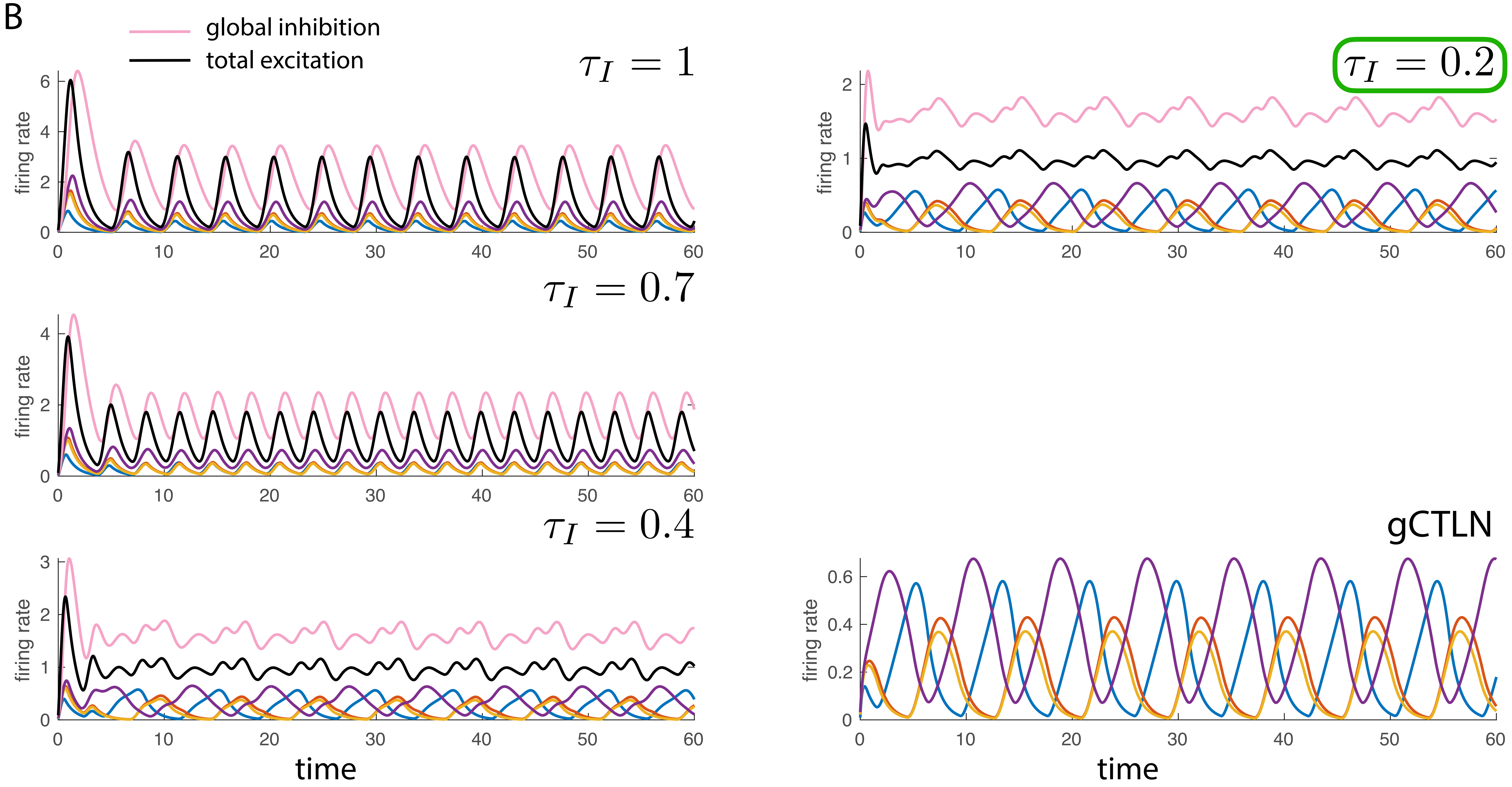
Beyond fixed points: do E-I TLNs produce similar dynamics to gCTLNs?



W for E-I TLN

$$W = \begin{pmatrix} 0 & 0 & 0 & a_4 & -1 \\ a_1 & 0 & a_3 & 0 & -1 \\ a_1 & a_2 & 0 & 0 & -1 \\ 0 & a_2 & a_3 & 0 & -1 \\ c_1 & c_2 & c_3 & c_4 & 0 \end{pmatrix}$$





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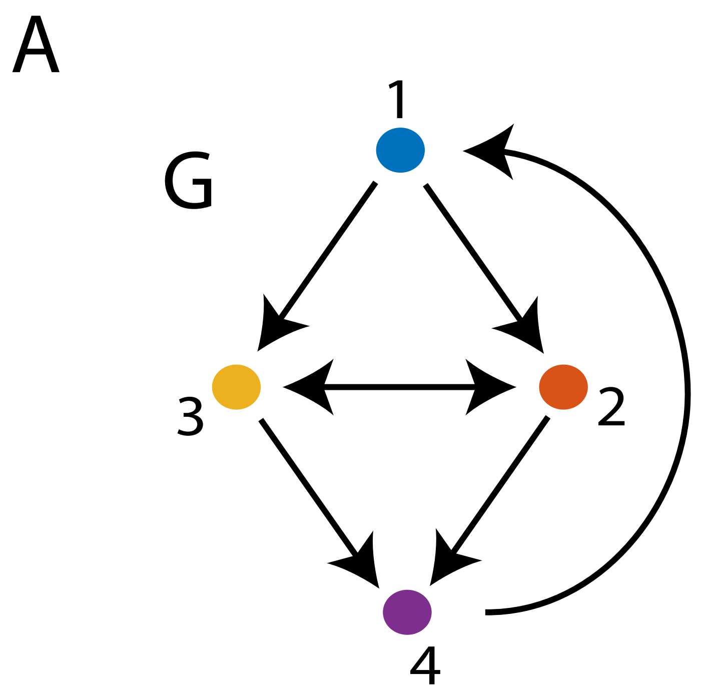

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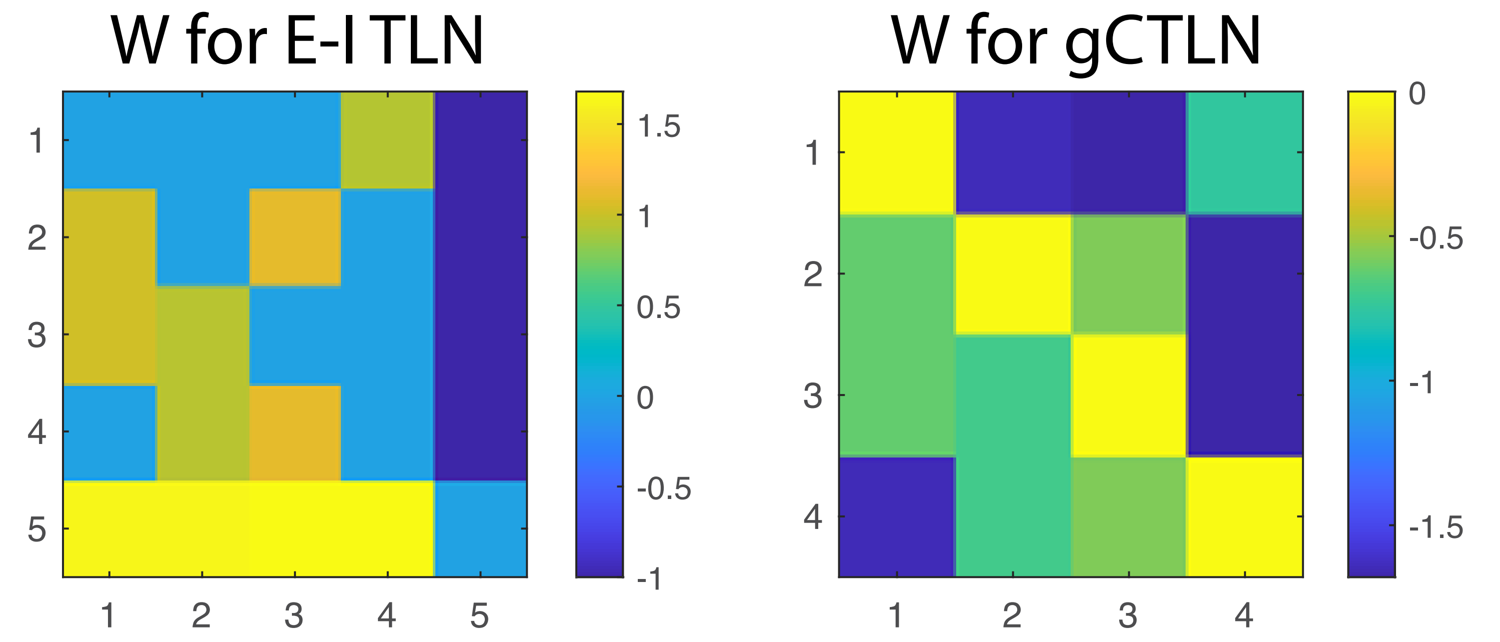
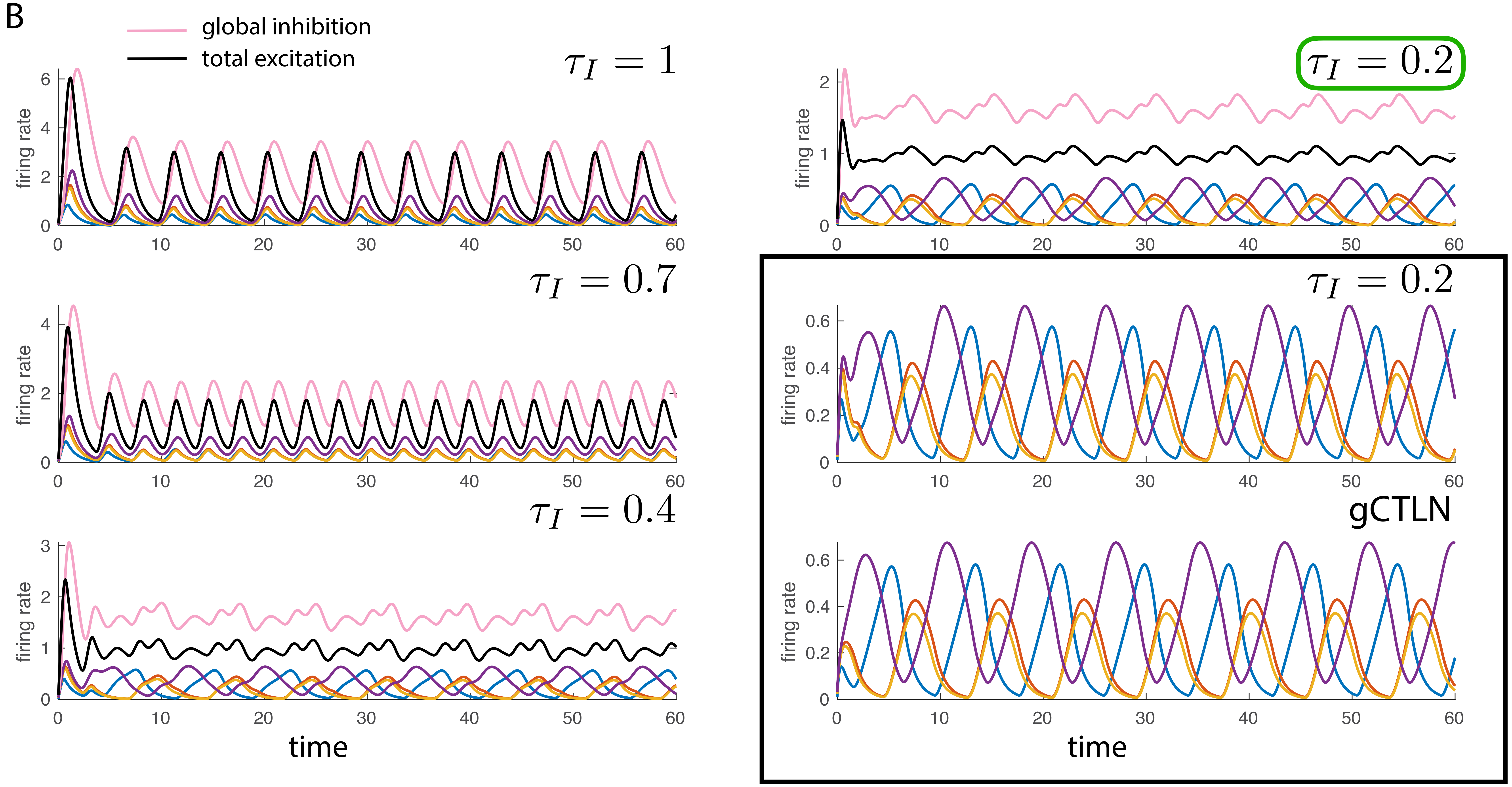
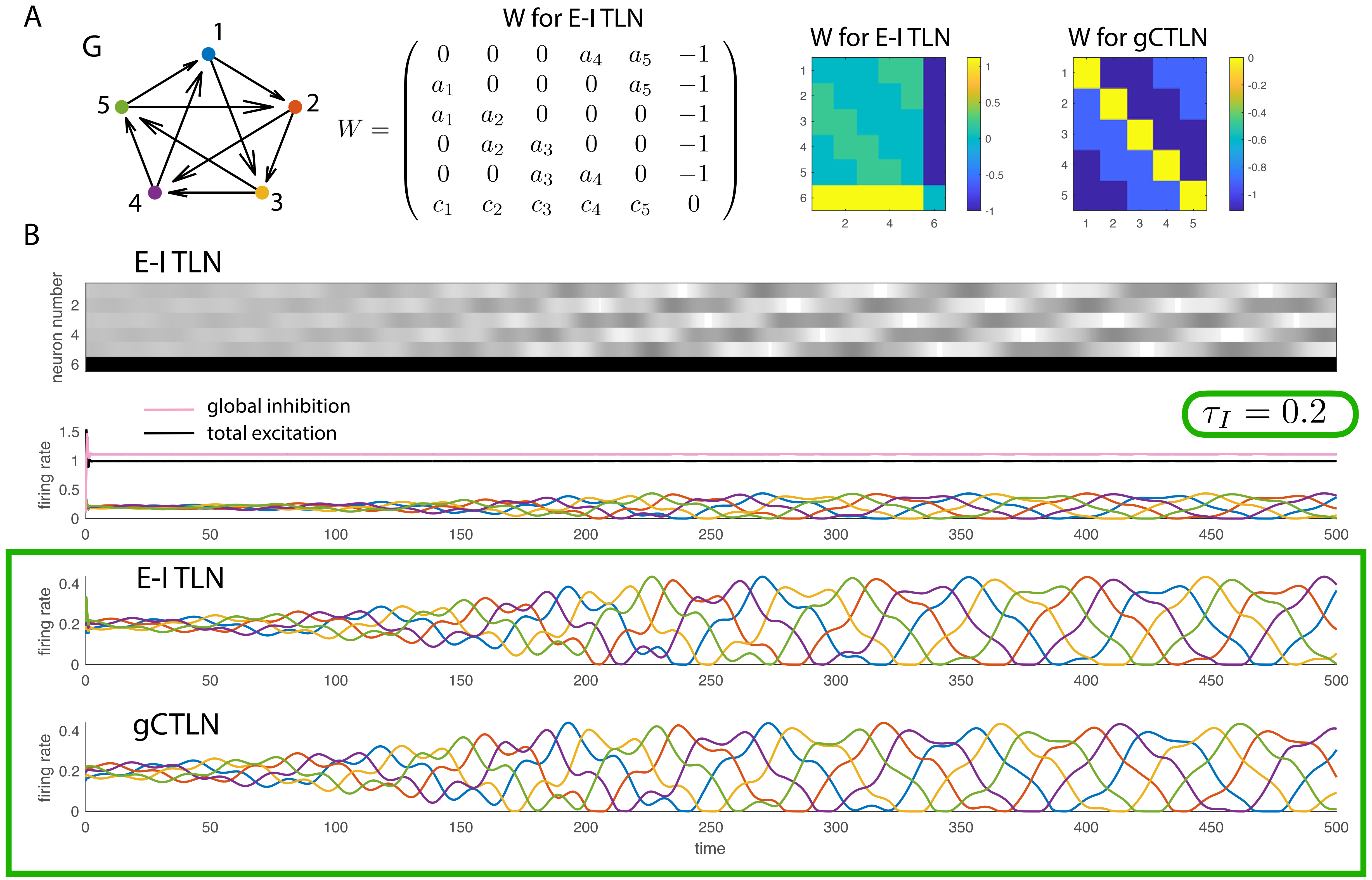
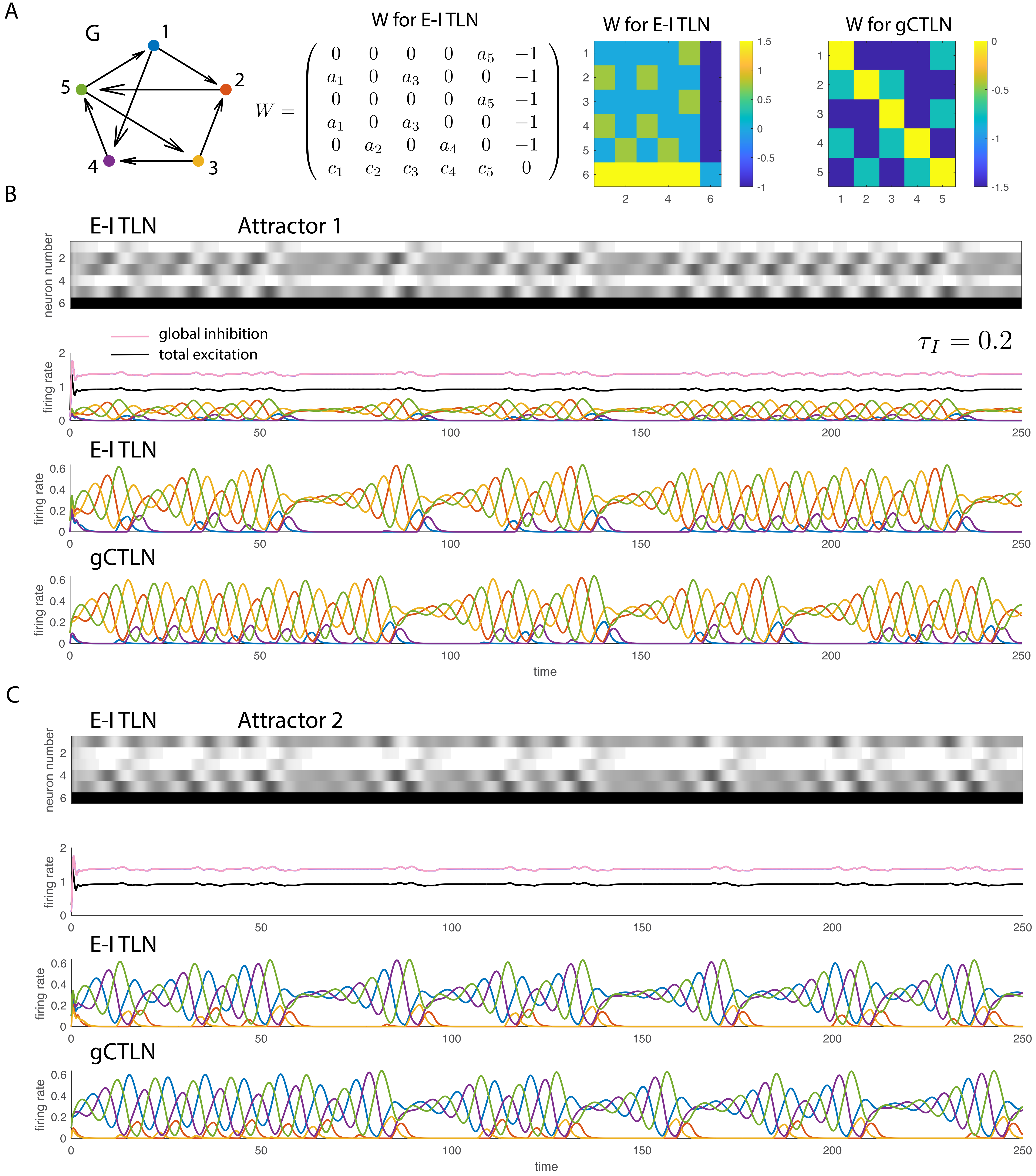




Even “exotic” attractors like Gaudi and baby chaos look the same



$$W = \begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 & -1 \\ a_1 & 0 & 0 & 0 & a_5 & -1 \\ a_1 & a_2 & 0 & 0 & 0 & -1 \\ 0 & a_2 & a_3 & 0 & 0 & -1 \\ 0 & 0 & a_3 & a_4 & 0 & -1 \\ c_1 & c_2 & c_3 & c_4 & c_5 & 0 \end{pmatrix}$$



$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & a_5 & -1 \\ a_1 & 0 & a_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & a_5 & -1 \\ a_1 & 0 & a_3 & 0 & 0 & -1 \\ 0 & a_2 & 0 & a_4 & 0 & -1 \\ c_1 & c_2 & c_3 & c_4 & c_5 & 0 \end{pmatrix}$$

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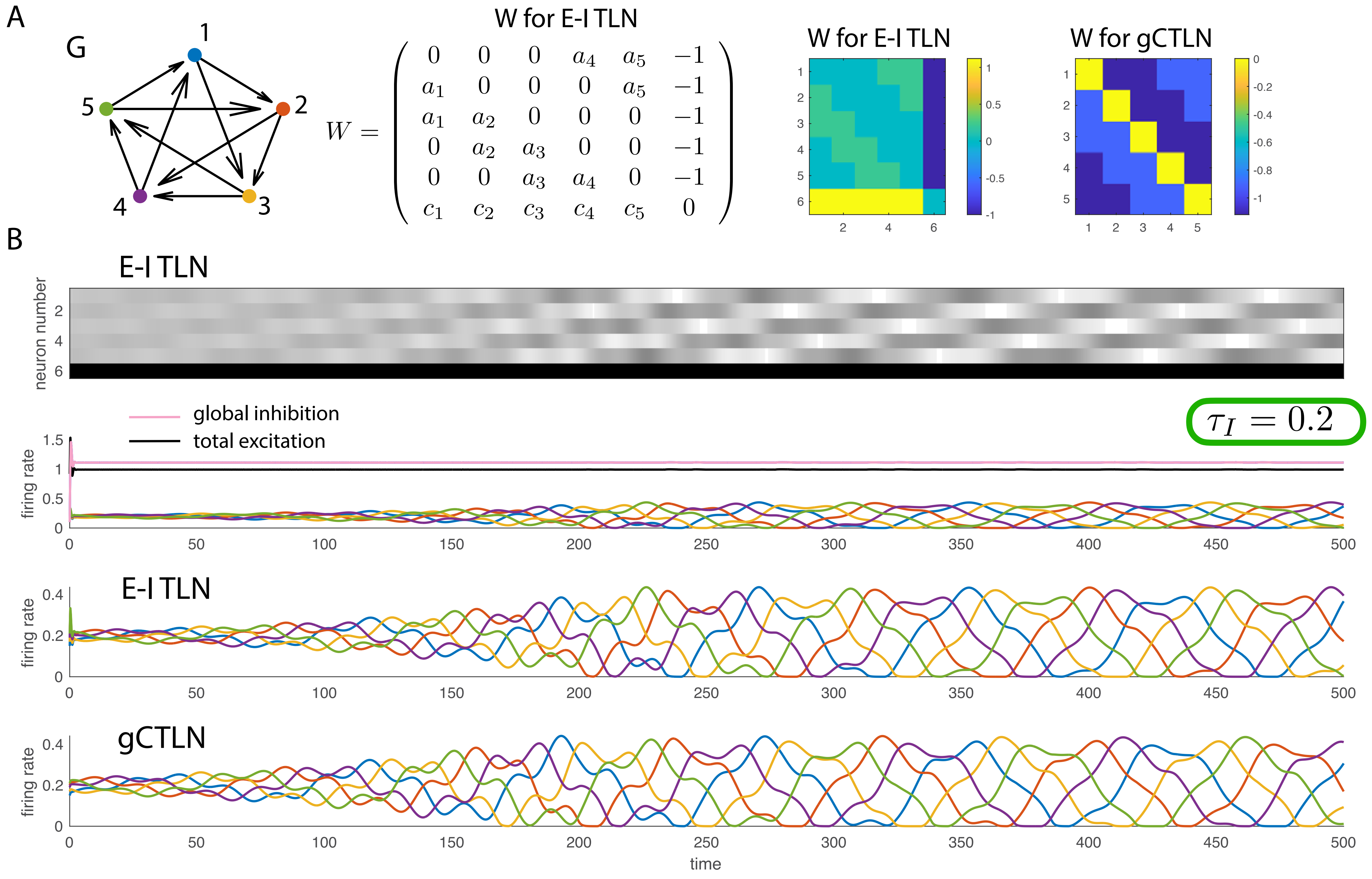
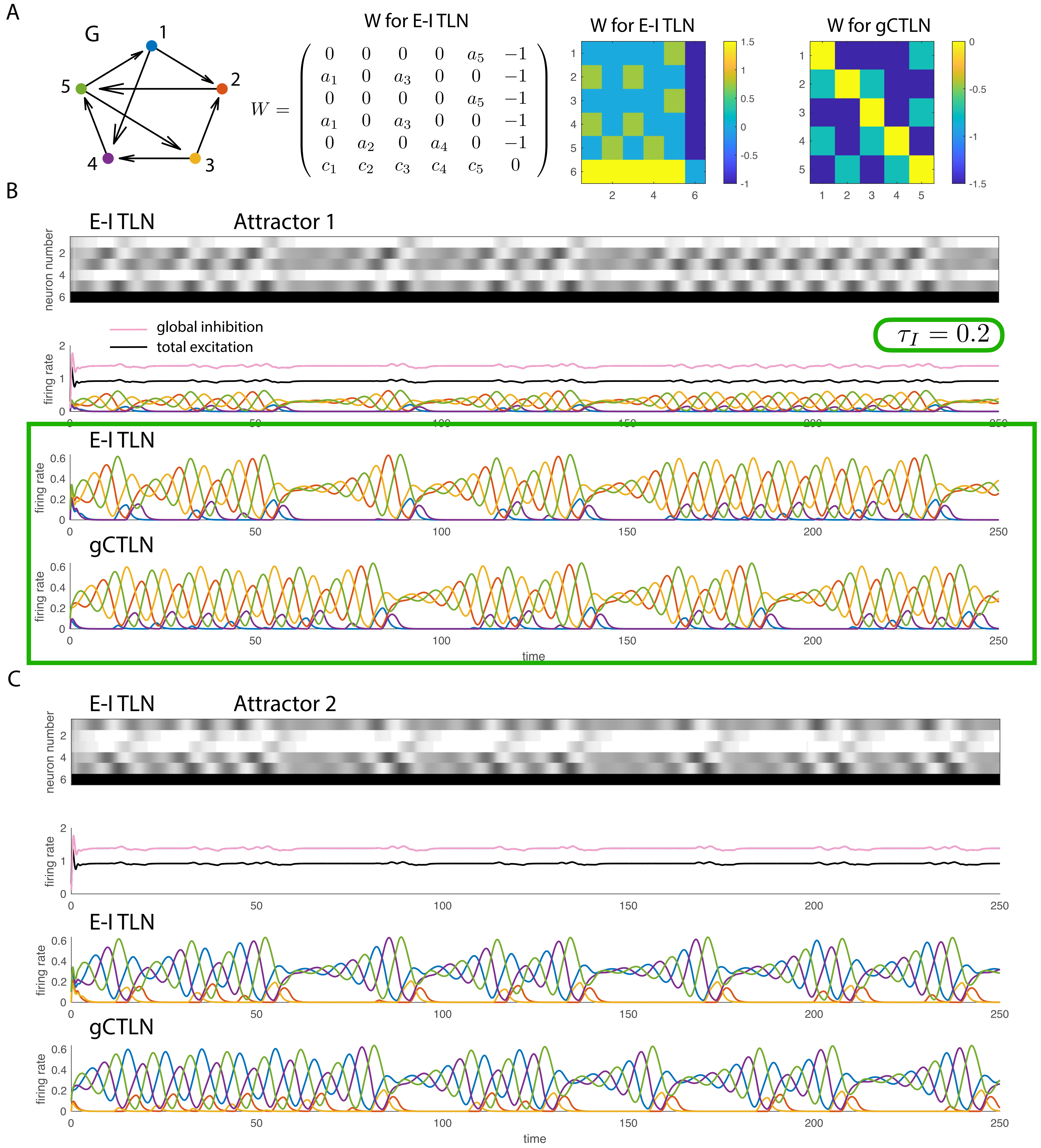


$$W = \begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 & -1 \\ a_1 & 0 & 0 & 0 & a_5 & -1 \\ a_1 & a_2 & 0 & 0 & 0 & -1 \\ 0 & a_2 & a_3 & 0 & 0 & -1 \\ 0 & 0 & a_3 & a_4 & 0 & -1 \\ c_1 & c_2 & c_3 & c_4 & c_5 & 0 \end{pmatrix}$$



$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & a_5 & -1 \\ a_1 & 0 & a_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & a_5 & -1 \\ a_1 & 0 & a_3 & 0 & 0 & -1 \\ 0 & a_2 & 0 & a_4 & 0 & -1 \\ c_1 & c_2 & c_3 & c_4 & c_5 & 0 \end{pmatrix}$$

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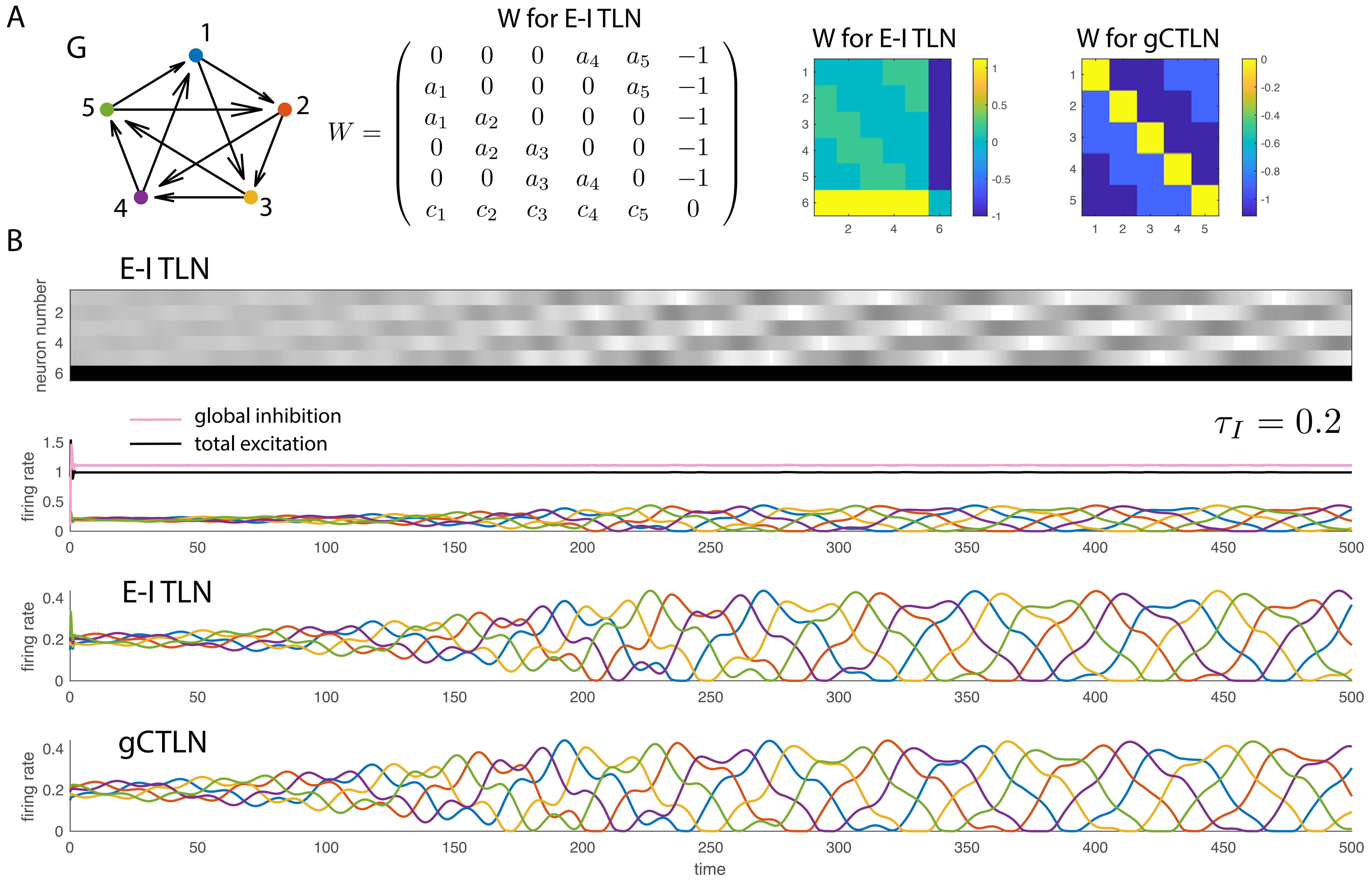
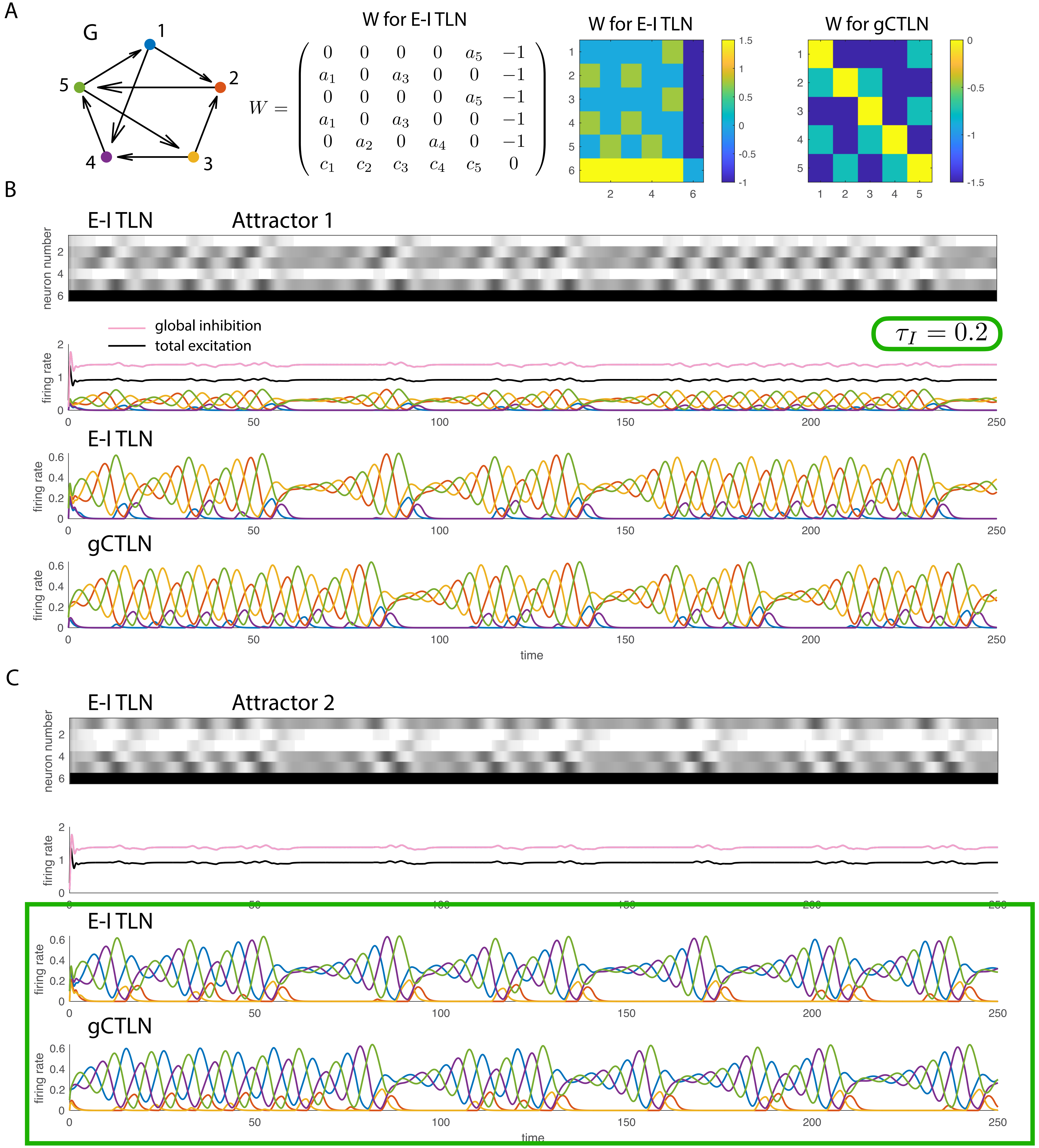


$$W = \begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 & -1 \\ a_1 & 0 & 0 & 0 & a_5 & -1 \\ a_1 & a_2 & 0 & 0 & 0 & -1 \\ 0 & a_2 & a_3 & 0 & 0 & -1 \\ 0 & 0 & a_3 & a_4 & 0 & -1 \\ c_1 & c_2 & c_3 & c_4 & c_5 & 0 \end{pmatrix}$$



$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & a_5 & -1 \\ a_1 & 0 & a_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & a_5 & -1 \\ a_1 & 0 & a_3 & 0 & 0 & -1 \\ 0 & a_2 & 0 & a_4 & 0 & -1 \\ c_1 & c_2 & c_3 & c_4 & c_5 & 0 \end{pmatrix}$$

Domination Theorems

Since E-I TLNs map to gCTLNs with the same fixed points, the domination theorems hold for E-I TLNs, too!

Theorem 1 (2024)

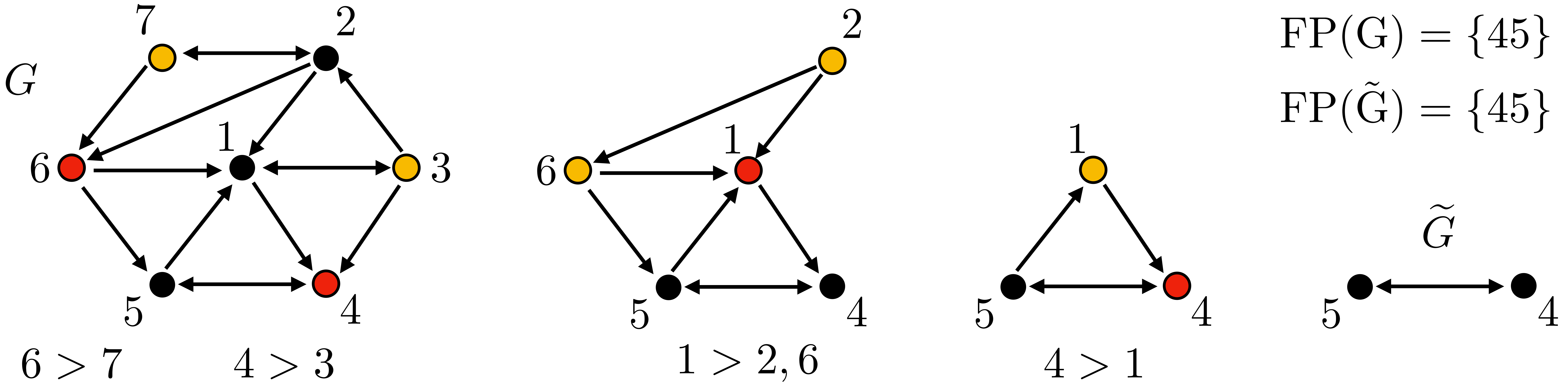
If j is a dominated node in G, then it drops out!

I.e., in any gCTLN, we have: $\mathrm{FP}(G) = \mathrm{FP}(G|_{[n]\setminus j})$

Theorem 2 (2024)

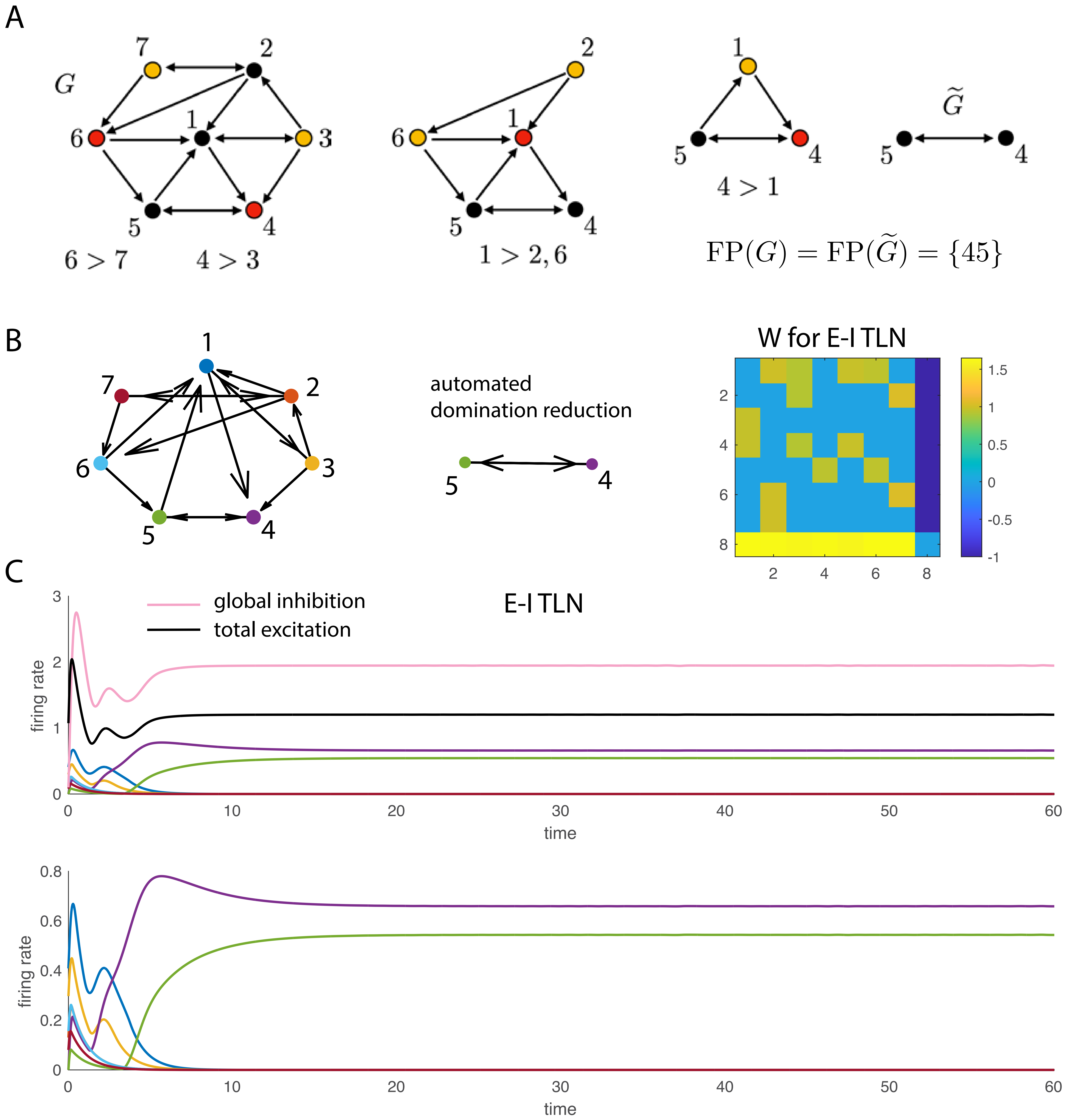
By iteratively removing dominated nodes, the final reduced graph G-tilde is unique. Moreover, $\mathrm{FP}(G) = \mathrm{FP}(\widetilde{G})$

Example



$\mathrm{FP(G)} = \{45\}$

$\mathrm{FP(\tilde{G})} = \{45\}$



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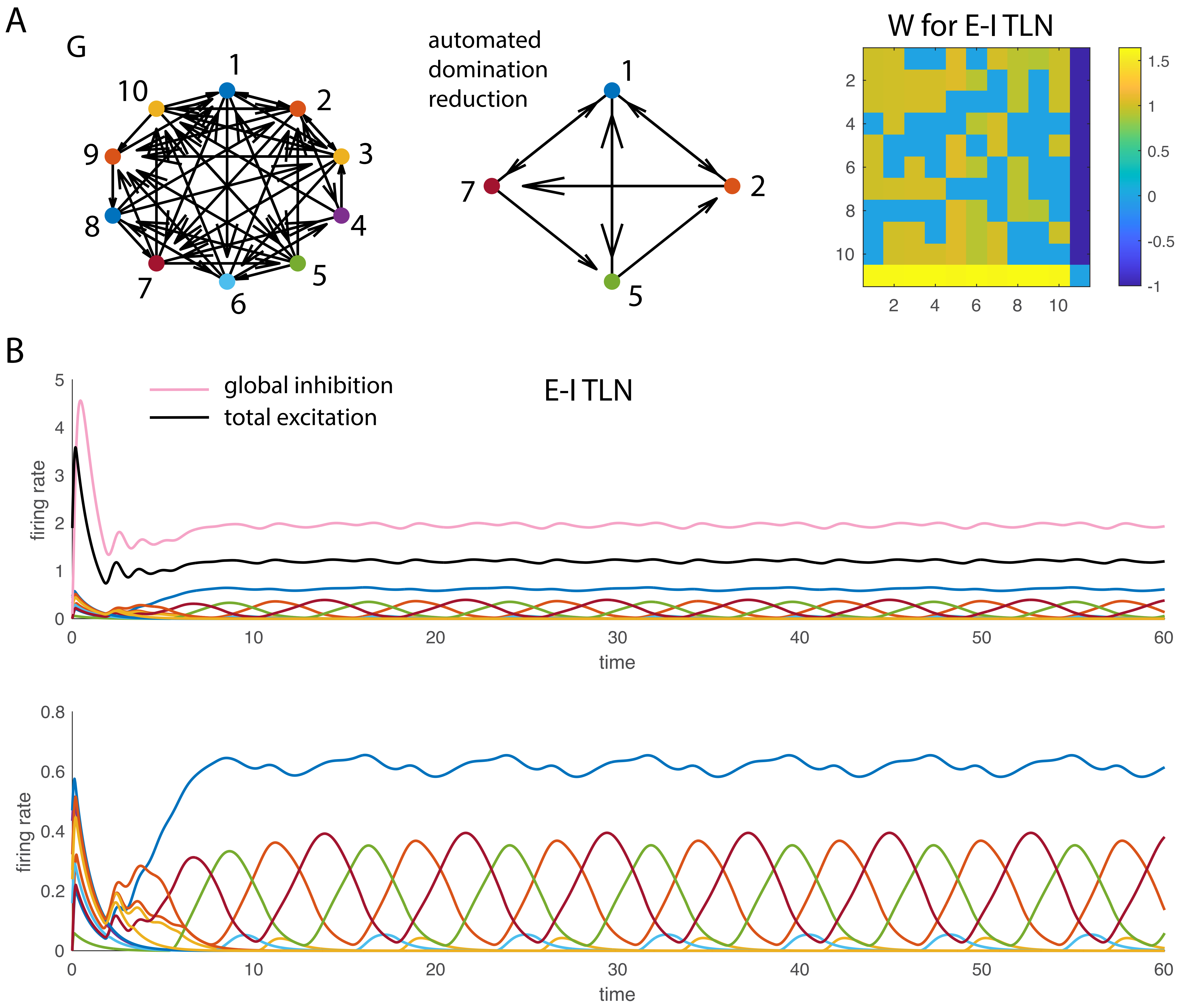
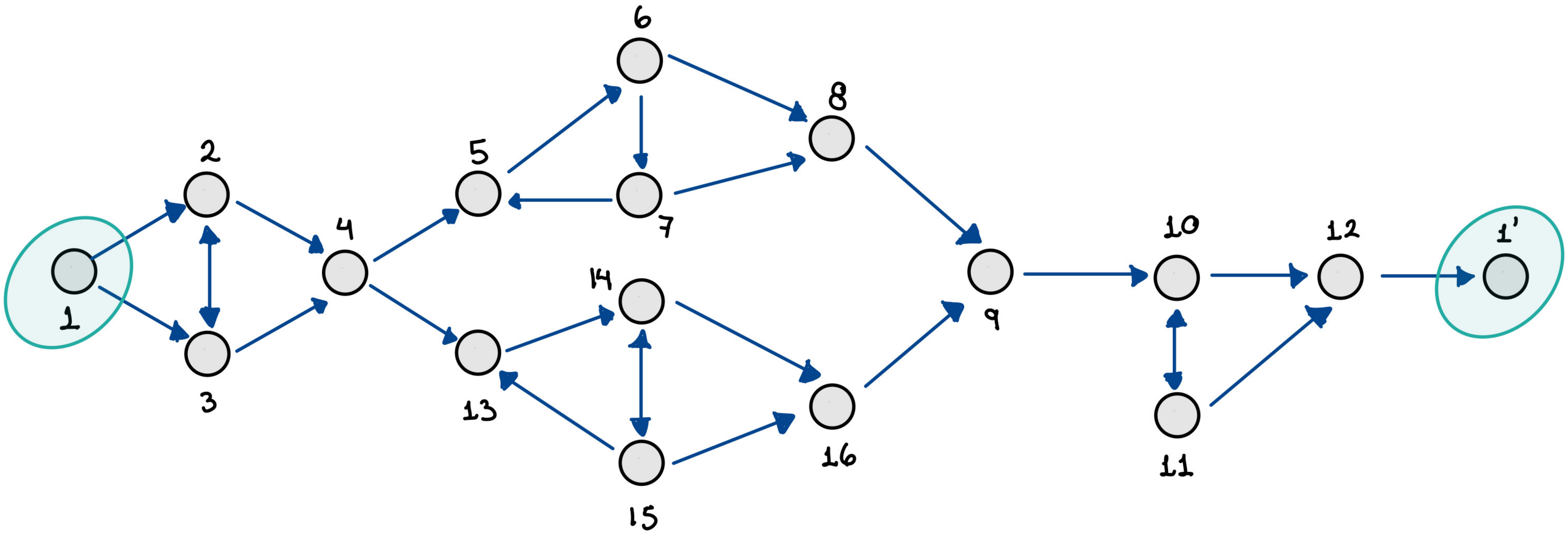


Since E-I TLNs map to gCTLNs with the same fixed points, the domination theorems hold for E-I TLNs, too!

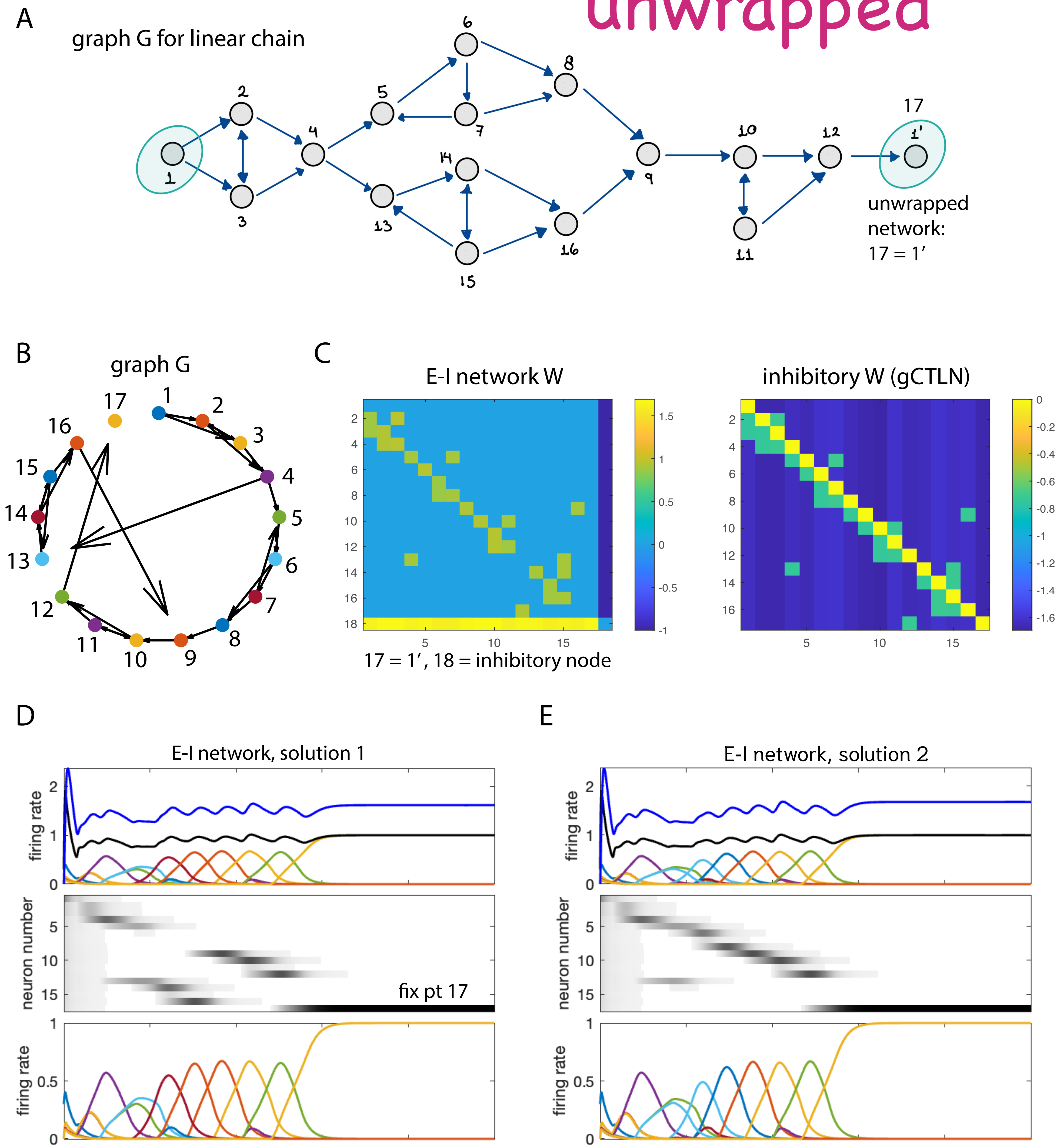
Cyclic chain example



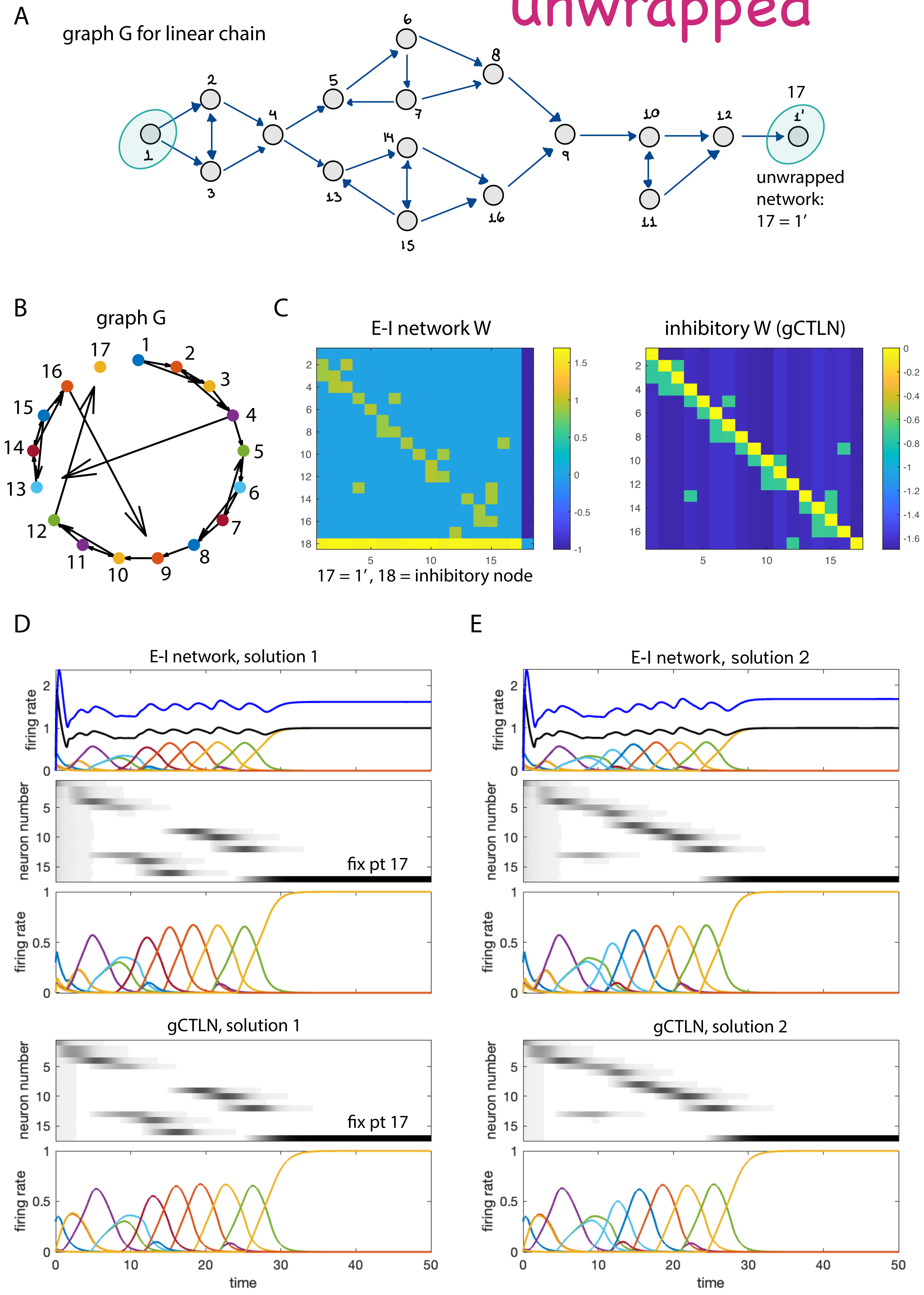
Identify $1 \equiv 1'$ at the end

Domination reduction cannot be done, and the network activity will loop around.

unwrapped

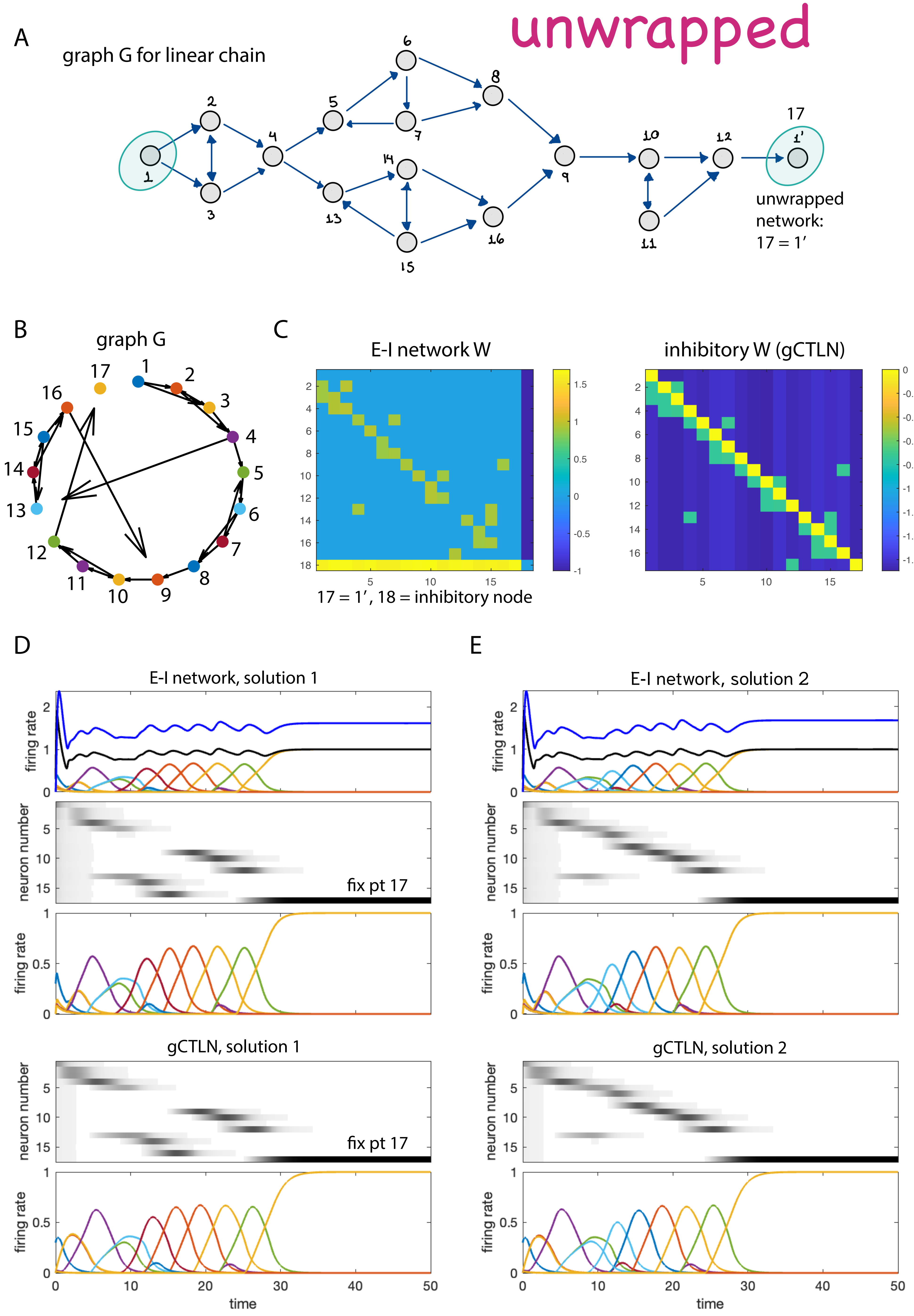

A
graph G for linear chain
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
1'
unwrapped network: 17 = 1'
B
graph G
C
E-I network W
17 = 1', 18 = inhibitory node
inhibitory W (gCTLN)
D
E-I network, solution 1
firing rate
neuron number
fix pt 17
E
E-I network, solution 2

unwrapped

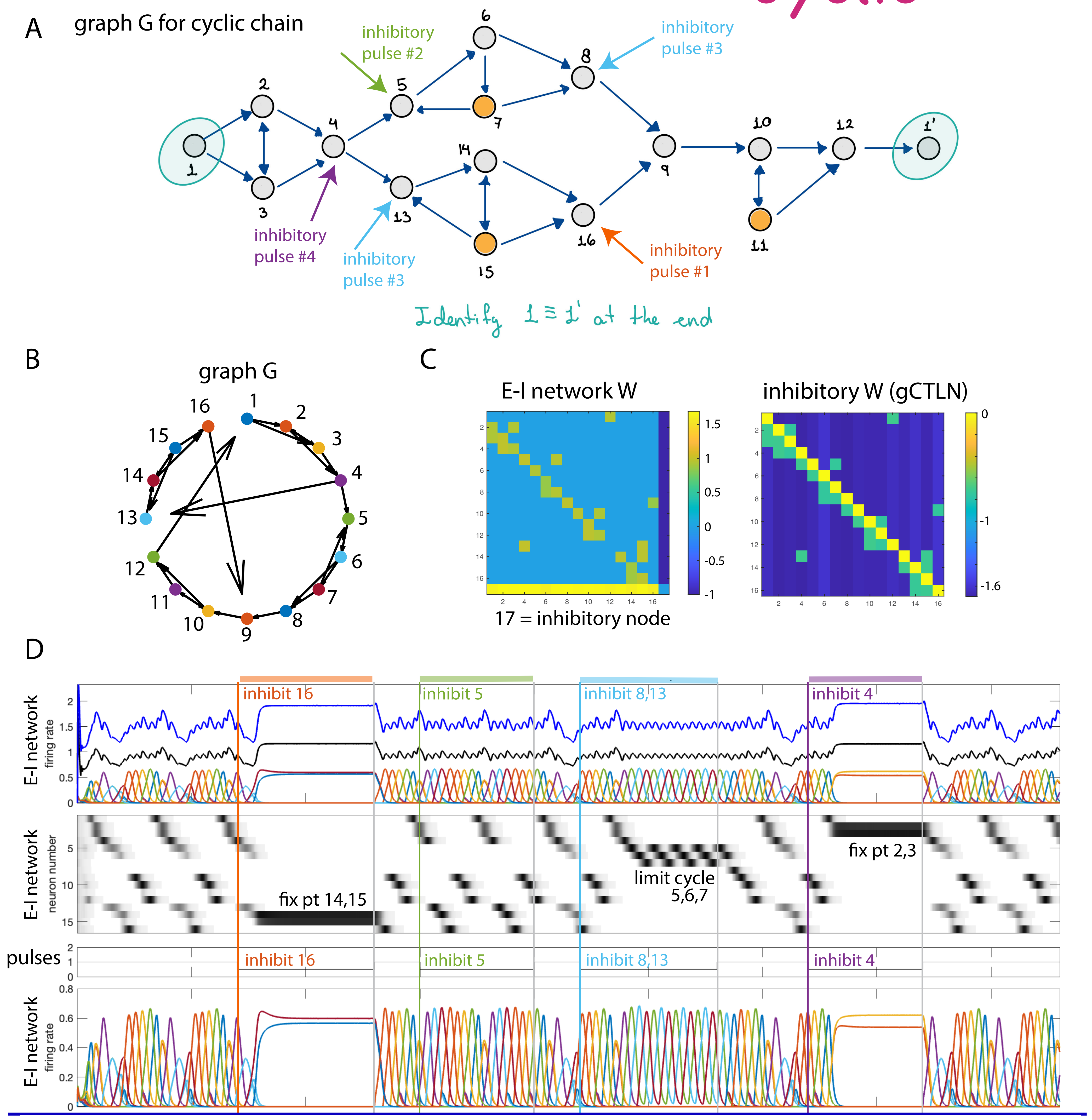


A
graph G for linear chain
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
1'
unwrapped
network:
17 = 1'
B
graph G
C
E-I network W
17 = 1', 18 = inhibitory node
inhibitory W (gCTLN)
D
E-I network, solution 1
gCTLN, solution 1
fix pt 17
E
E-I network, solution 2
gCTLN, solution 2
firing rate
neuron number
time

unwrapped

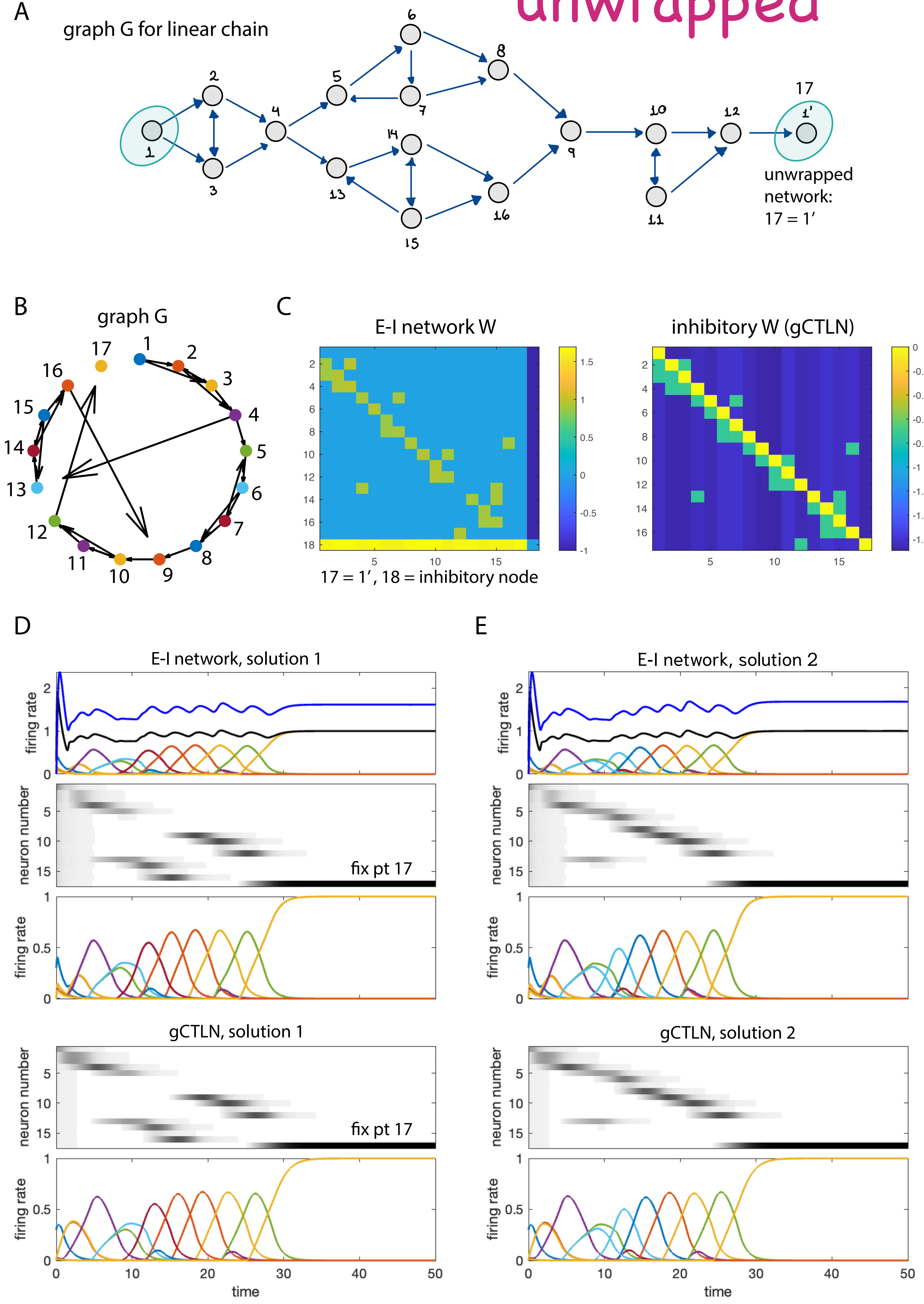


cyclic

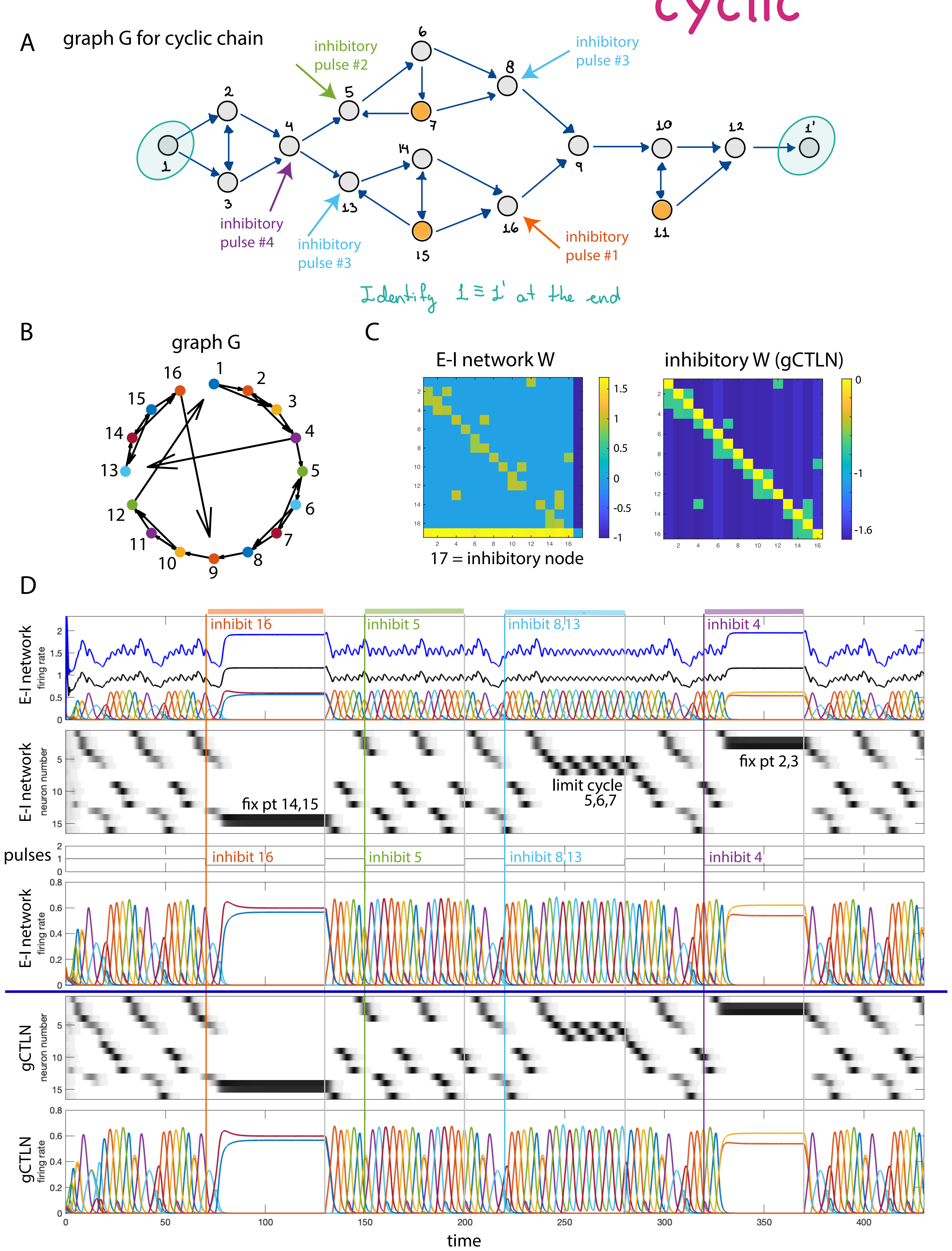


Identify 1 ≡ 1' at the end

unwrapped

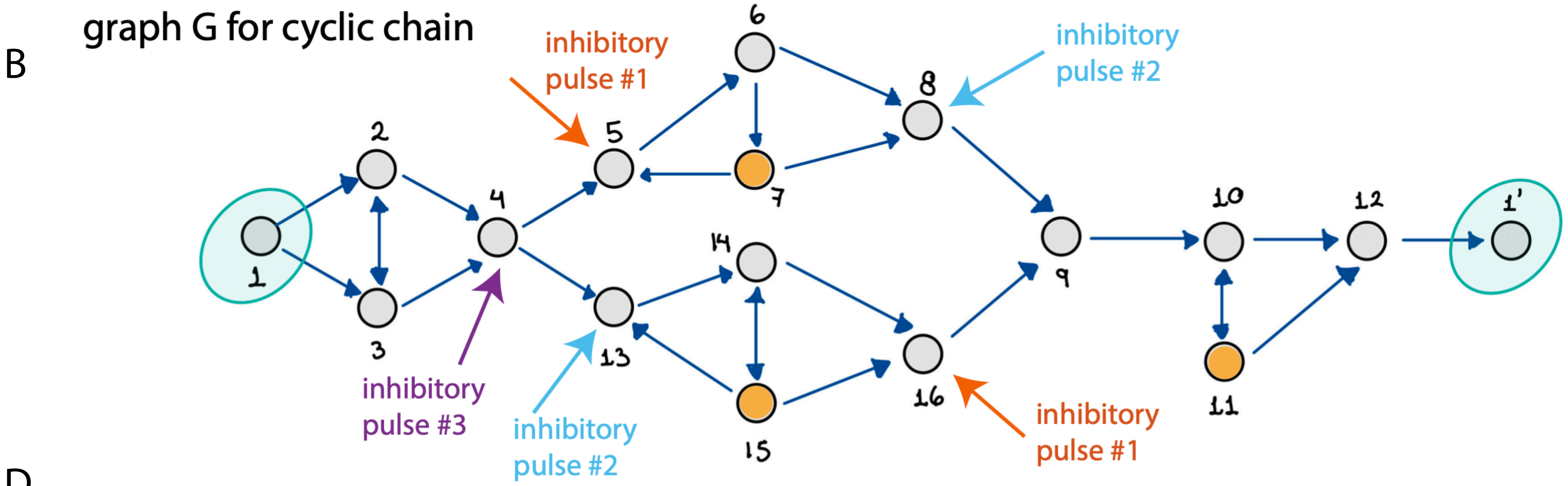


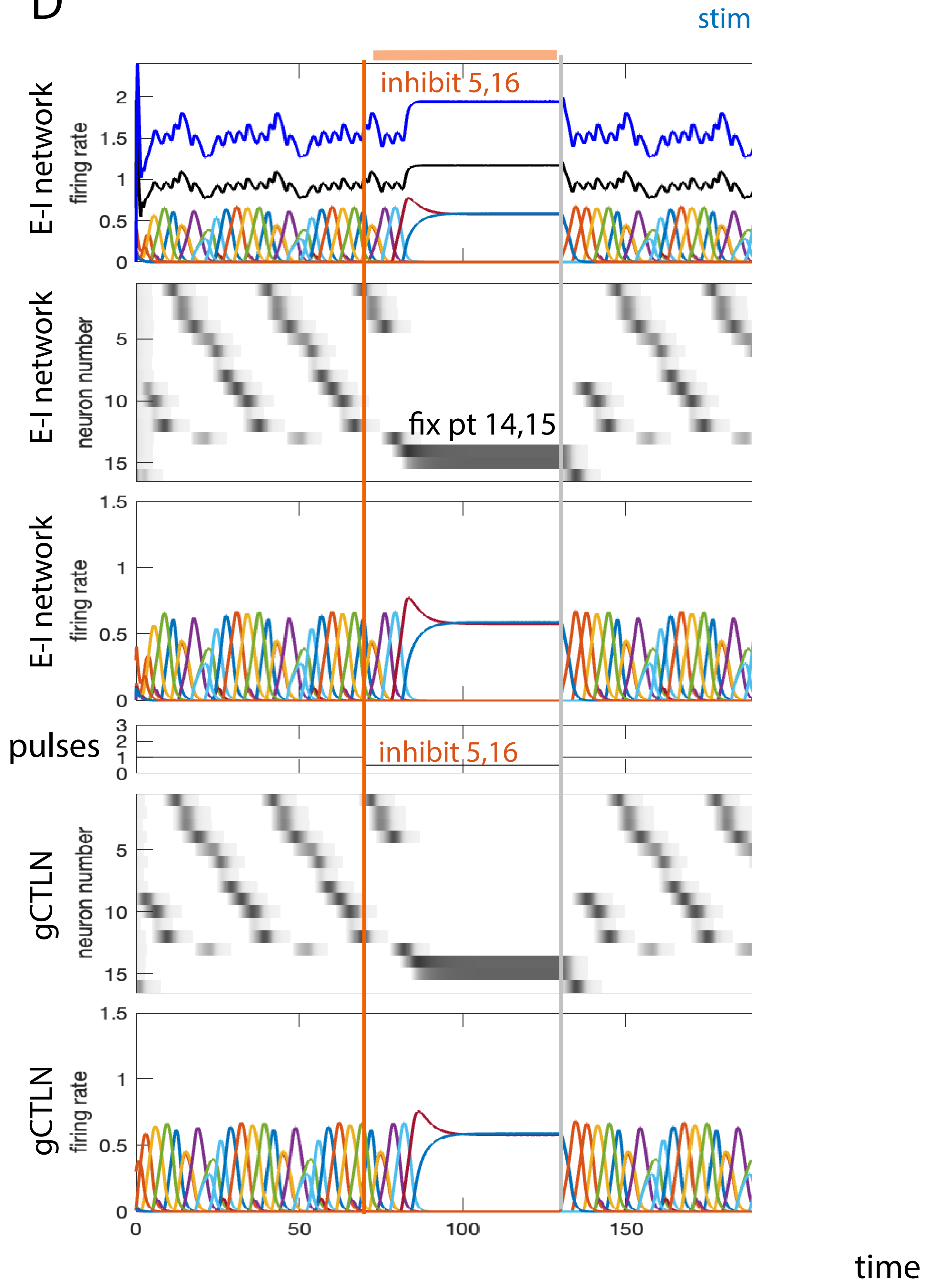
cyclic



Identify 1 ≡ 1' at the end

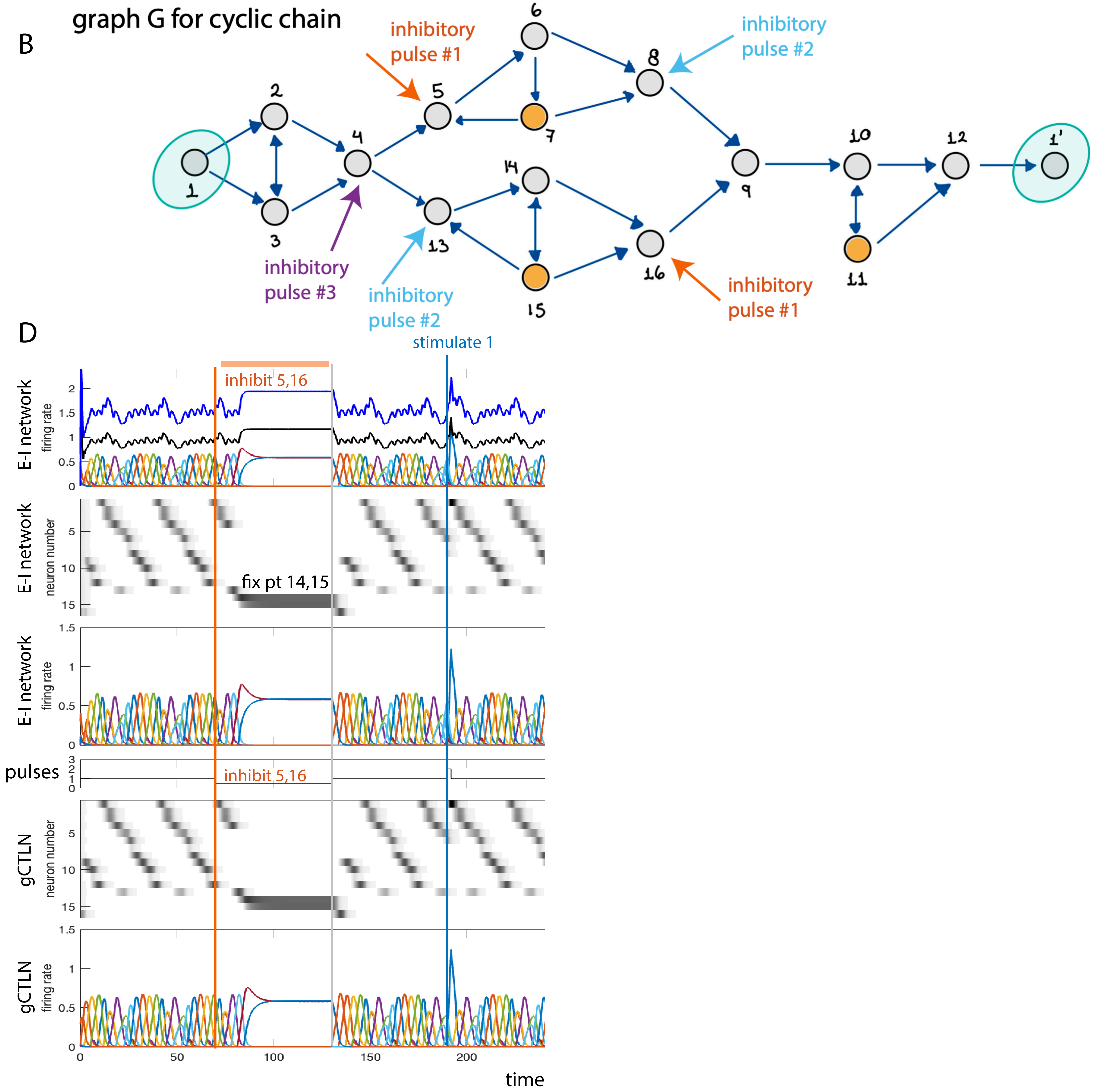
inhibitory pulses
= stop signs





inhibitory pulses
= stop signs

excitatory pulses
= teleportation



inhibitory pulses
= stop signs

excitatory pulses
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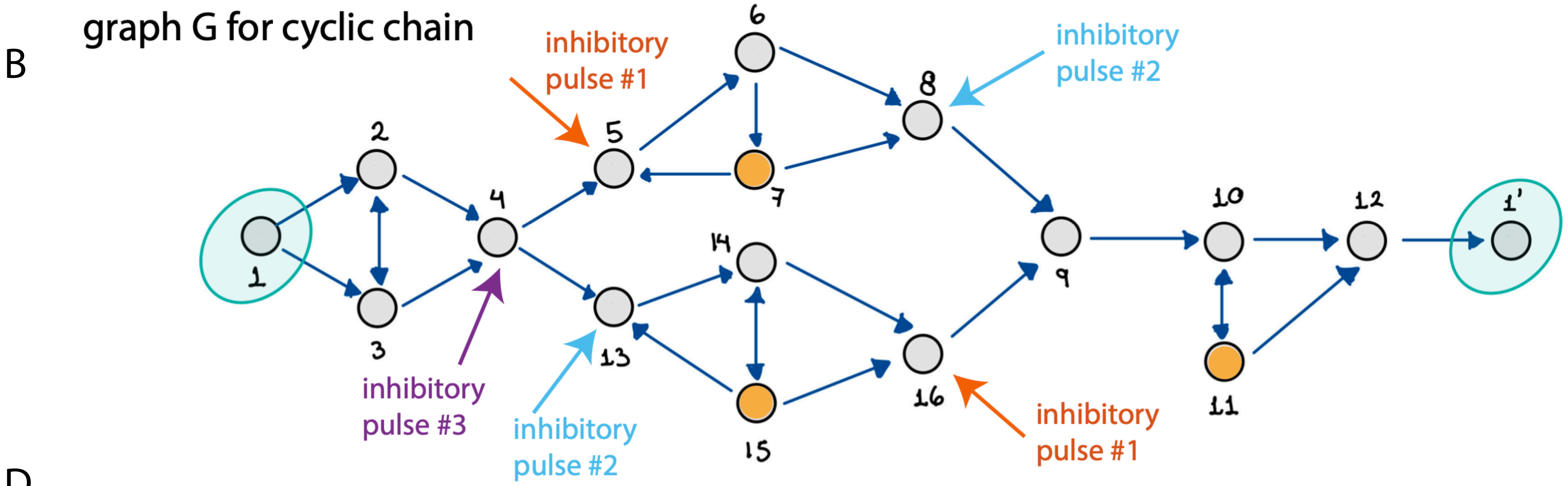

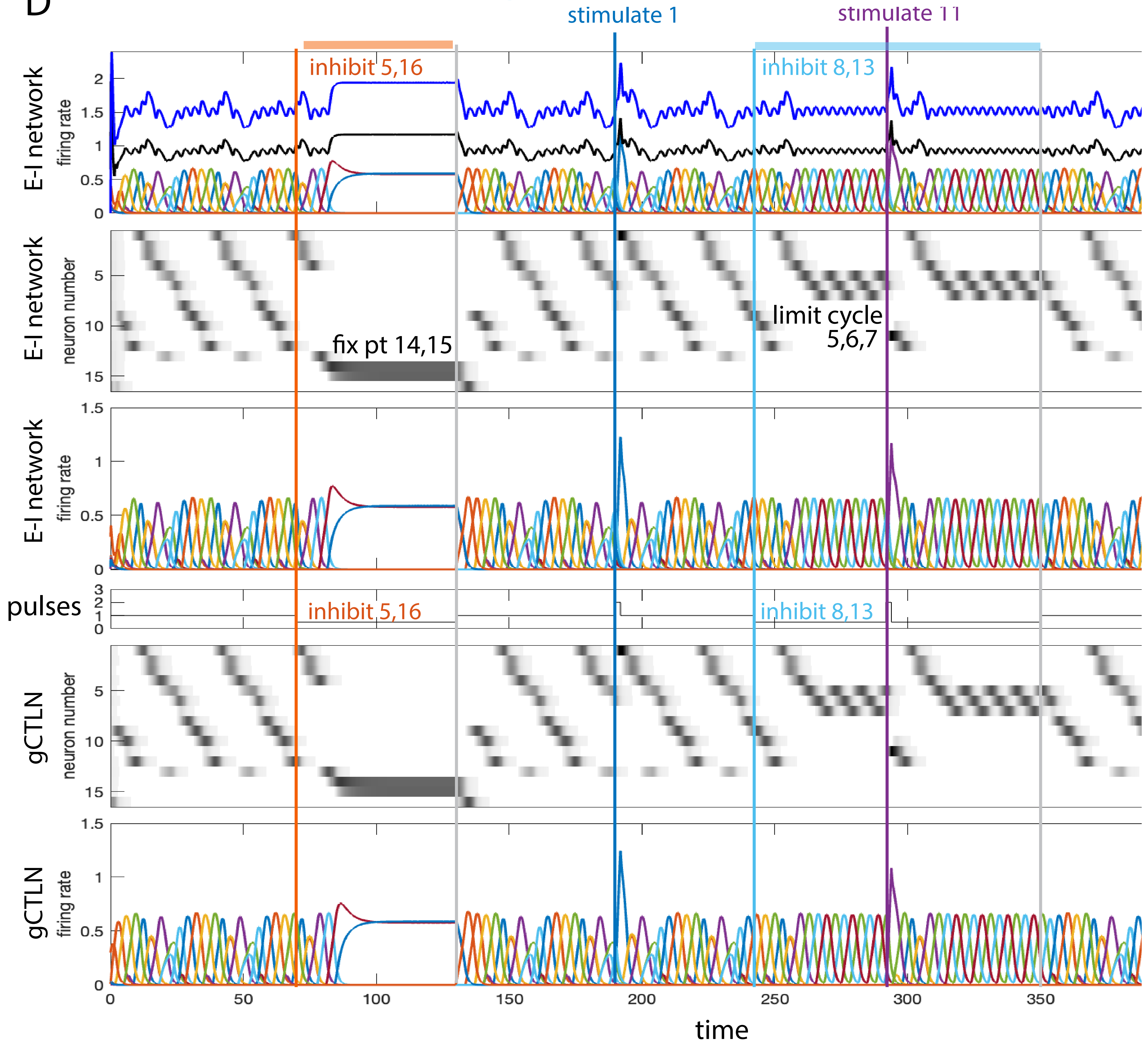

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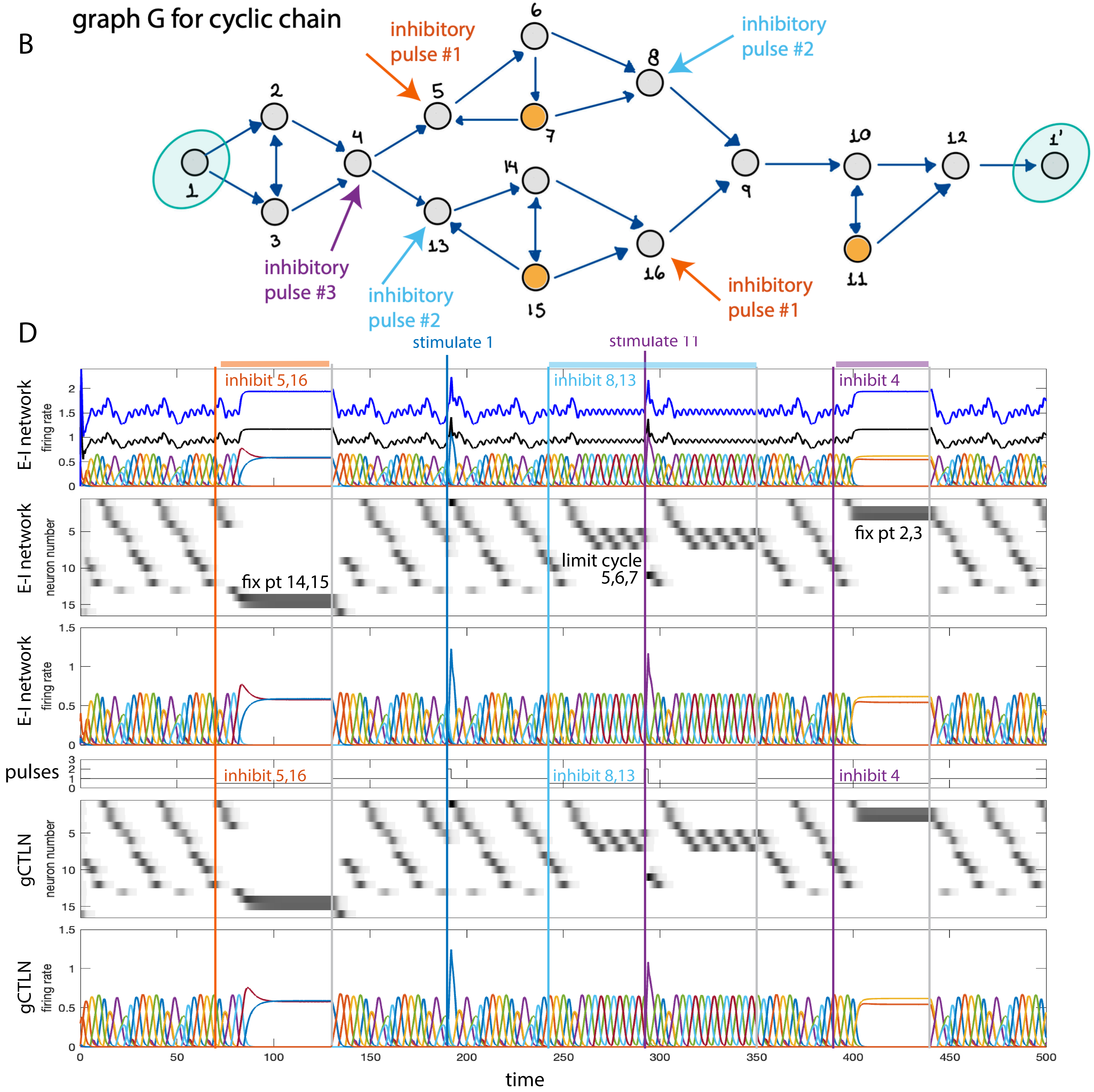


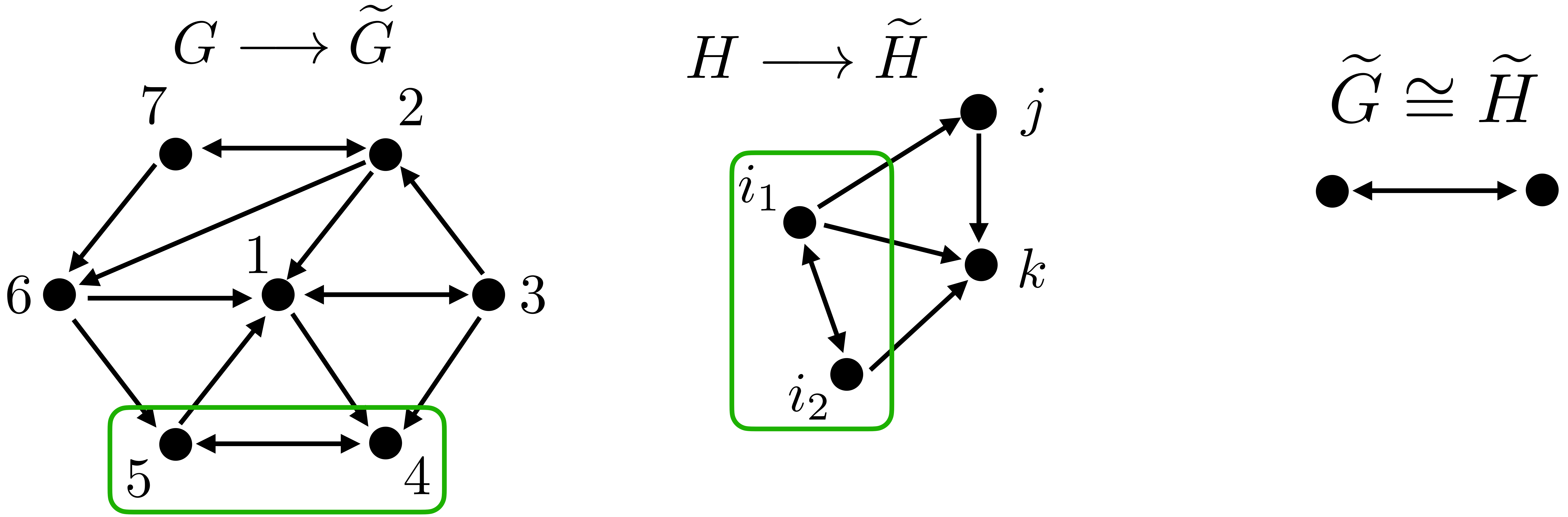
Plan of the talk

- Brief intro to TLNs, CTLNs, and gCTLNs
- Fixed points and attractors and graph rules
- Domination
- Dominoes and inhibitory control
- E-I TLNs
- Domination-reduction in connectomes

Can domination be useful for connectome analysis?

Every graph has a unique domination reduction: $G \longrightarrow \widetilde{G}$

Two graphs with the same reduction are in the same domination equivalence class.



$G \longrightarrow \widetilde{G}$

$H \longrightarrow \widetilde{H}$

$\widetilde{G} \cong \widetilde{H}$

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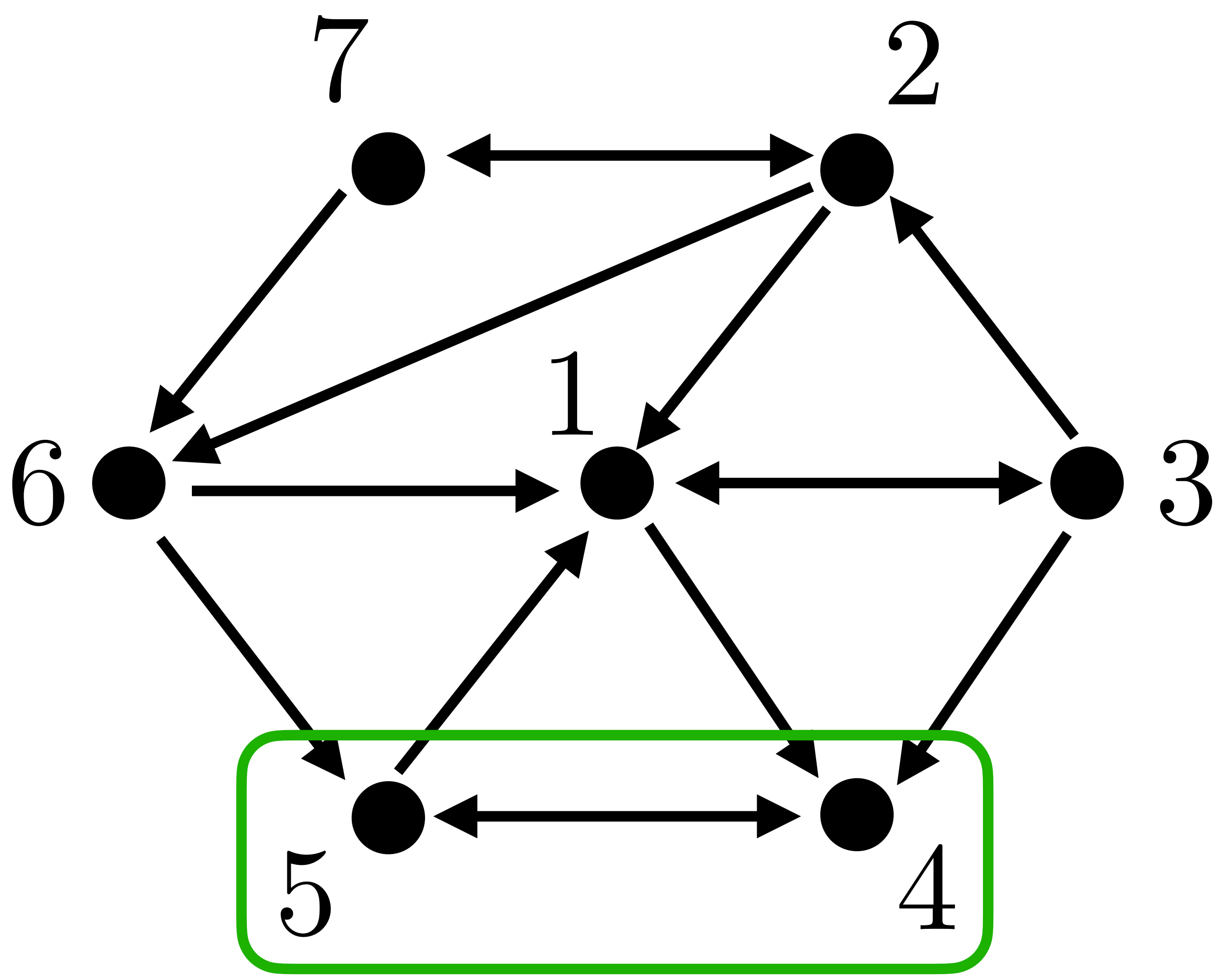
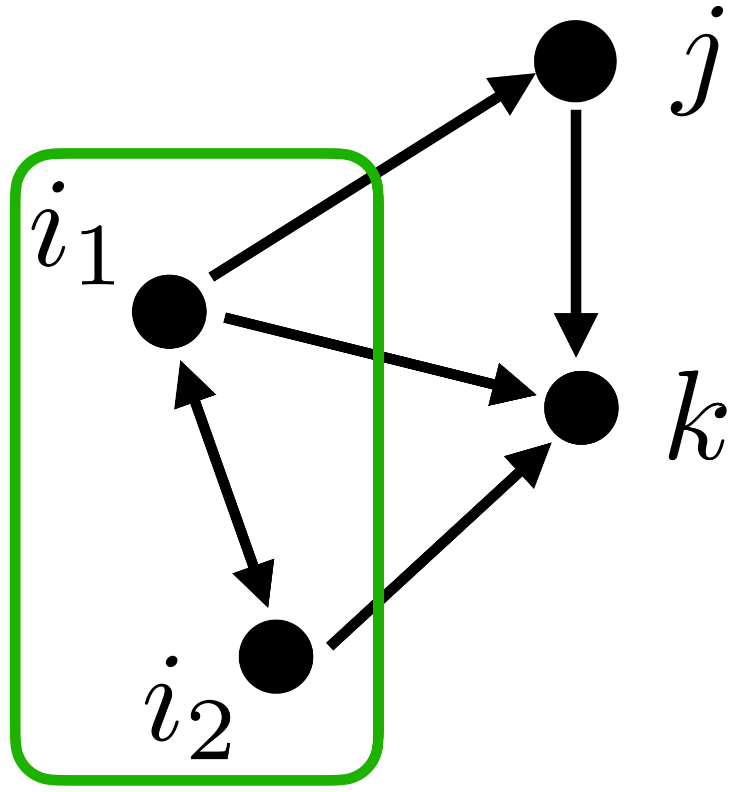


$H \longrightarrow \widetilde{H}$



$\widetilde{G} \cong \widetilde{H}$

1. Are overrepresented graphical motifs more likely to be reducible or irreducible?

2. Which motifs are domination-equivalent?

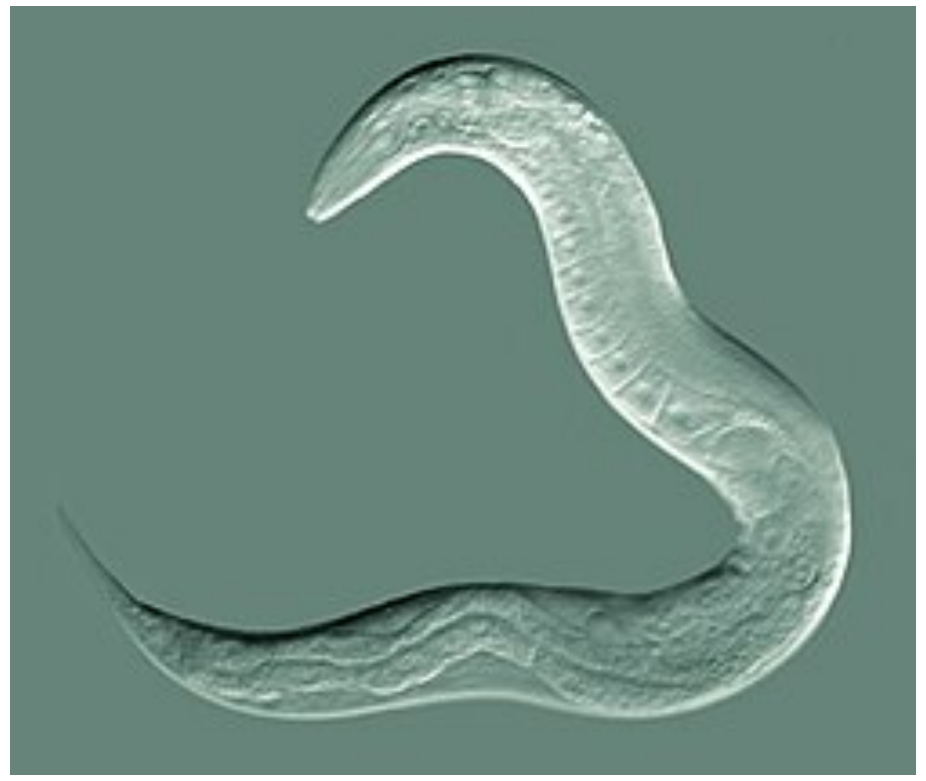
3. What about larger portions of the connectome: do they reduce via domination?

Very preliminary analysis

Graph motifs team at JHU

Jordan Matelsky (also at Penn)
Patricia Rivlin
Michael Robinette
Erik Johnson
Brock Wester

Johns Hopkins University Applied Physics Laboratory,
Research & Exploratory Development Department



C. elegans E-E network:

G has143 nodes

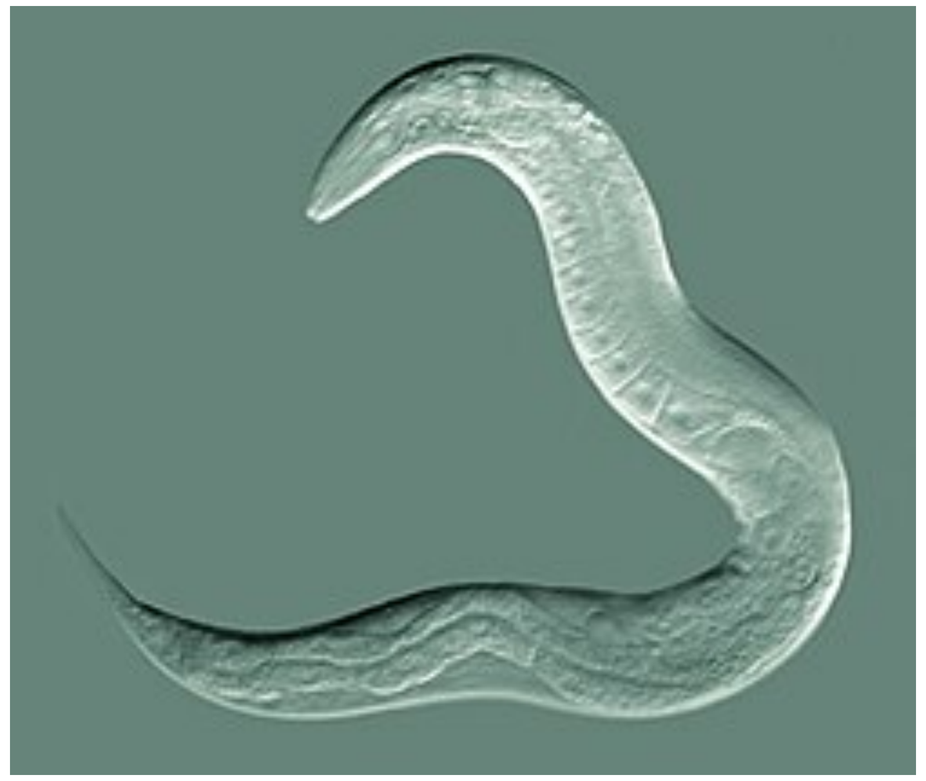
reduced G: 104 nodes

Joaquín Castañeda Castro

We first strip out everything but chemical synapses, then tag neurons by their small-molecule neurotransmitters—acetylcholine/glutamate as excitatory, GABA as inhibitory—next we grab the induced subgraph of neurons that fire ACh/Glu but no GABA. That's our 'excitatory' network. And yes—it's just a conservative, transmitter-based proxy for valence; real C. elegans synaptic polarity is far messier (receptors, modulators, co-transmission, gap junctions, etc.) All blame goes to Jordan Matelsky, Carina did nothing wrong.

Very preliminary analysis

Is a reduction from 143 -> 104 nodes common or rare in a random graph with matching edge probability?



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Very preliminary analysis

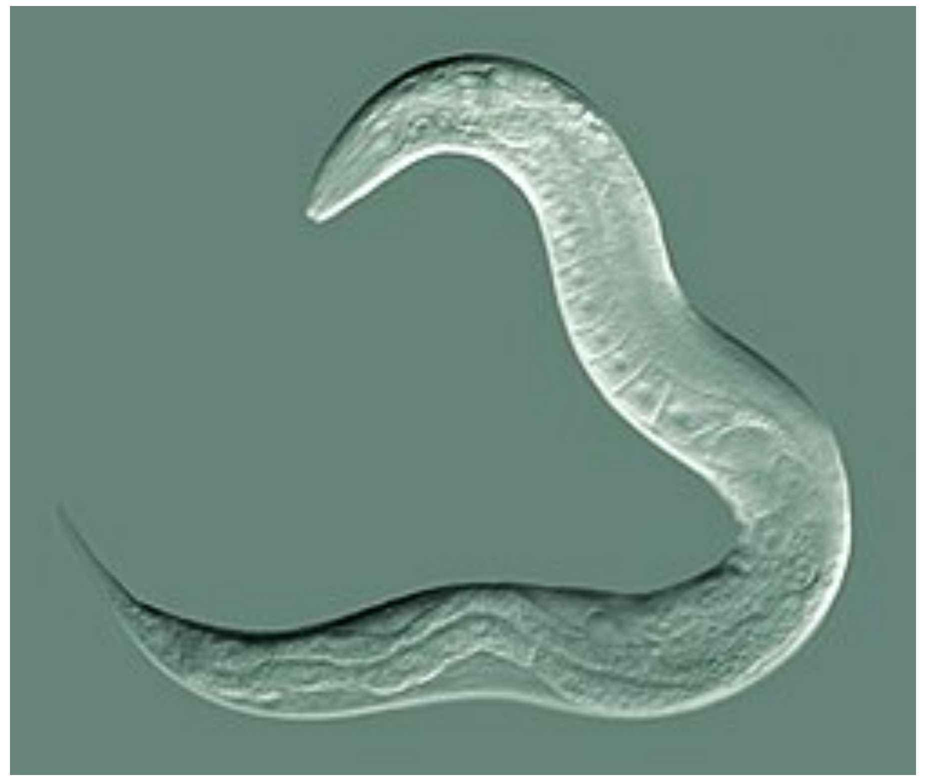
Is a reduction from 143 -> 104 nodes common or rare in a random graph with matching edge probability?

1 million E-R random graphs with matching p = 0.054

Distribution of domination reductions:

- 143 nodes: 782,590
- 142 nodes: 189,951
- 141 nodes: 24,951
- 140 nodes: 2,307
- 139 nodes: 185
- 138 nodes: 15
- 137 nodes: 1

VERY RARE!!



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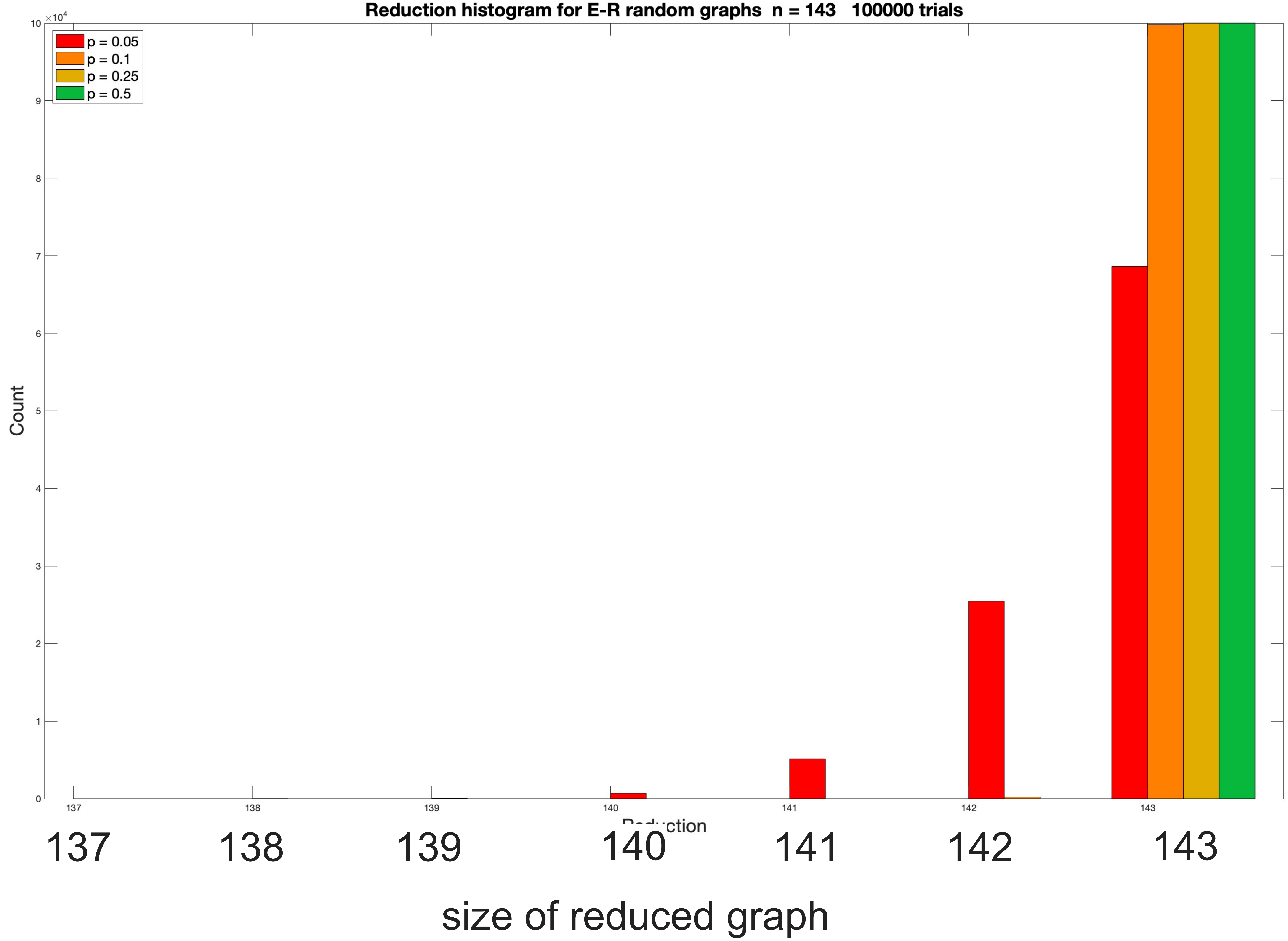
Reduction sizes of E-R random graphs of size n=143 with p = 0.05, 0.1, 0.25, 0.5



Back to our motivating questions and ideas:

1. The brain is a dynamical system. ("The brain is a computer.")
2. How does connectivity shape dynamics?
3. By studying ANNs that are dynamical systems, we can generate hypotheses about the dynamic meaning/role of various network motifs.
4. Network motifs can be composed as dynamic building blocks with predictable properties.
5. One network (by architecture/connectivity) is really many networks in the presence of neuromodulation or external control.

Domination is a graph property that comes out of the nonlinear dynamics, it is not something that graph theorists or network scientists were already paying attention to.

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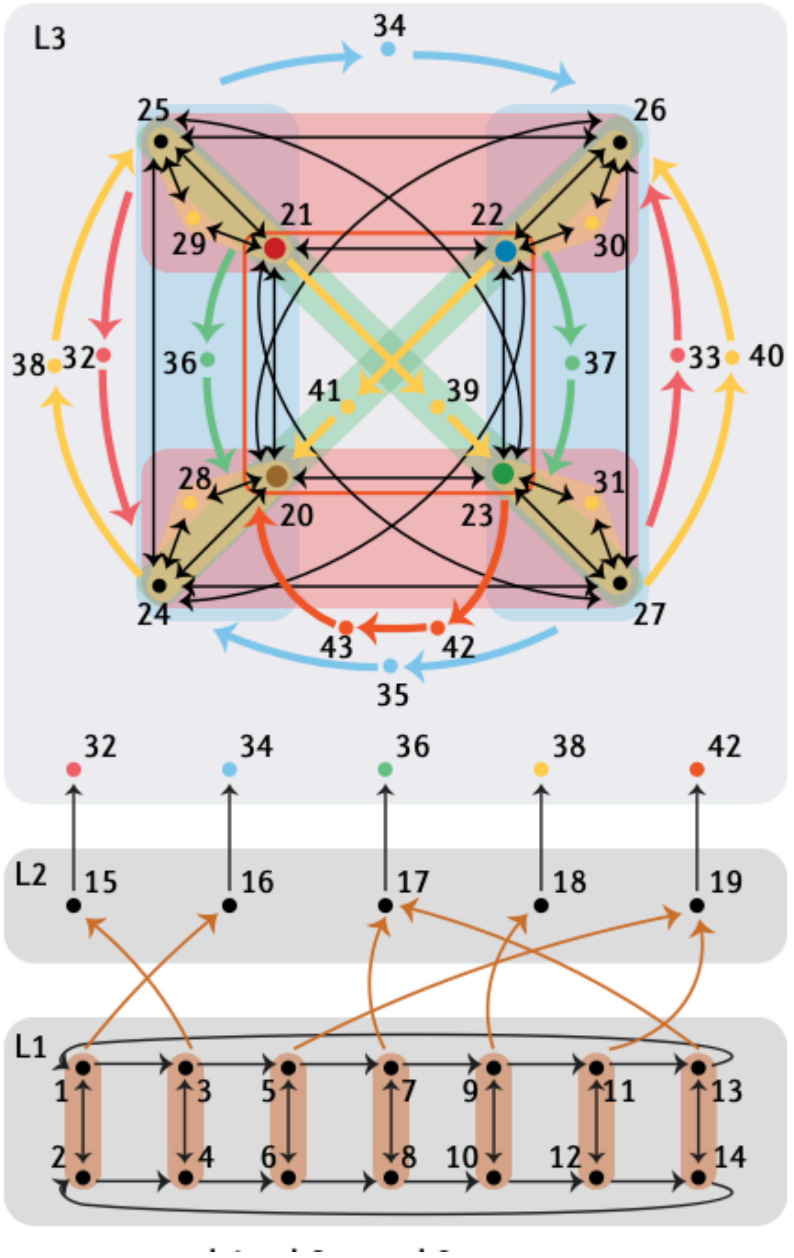
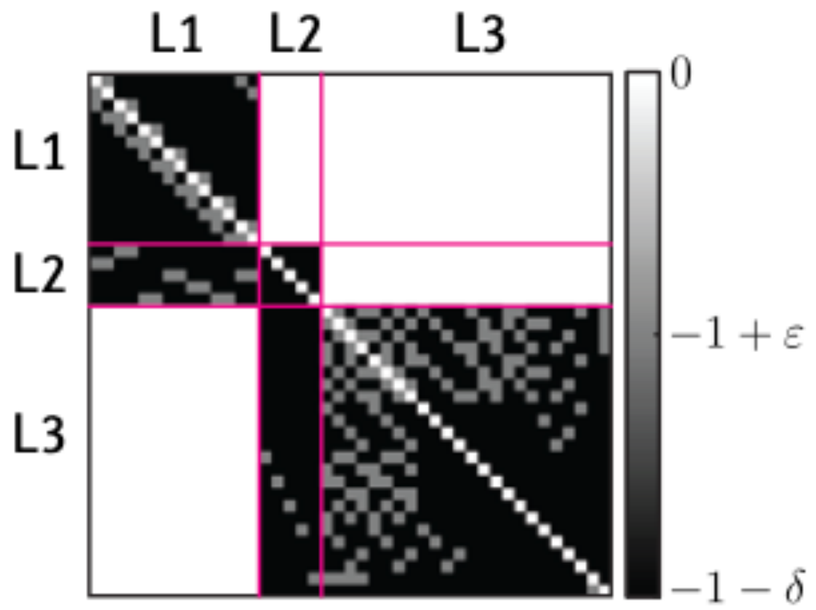
- Bonus: advertisement for some other related work

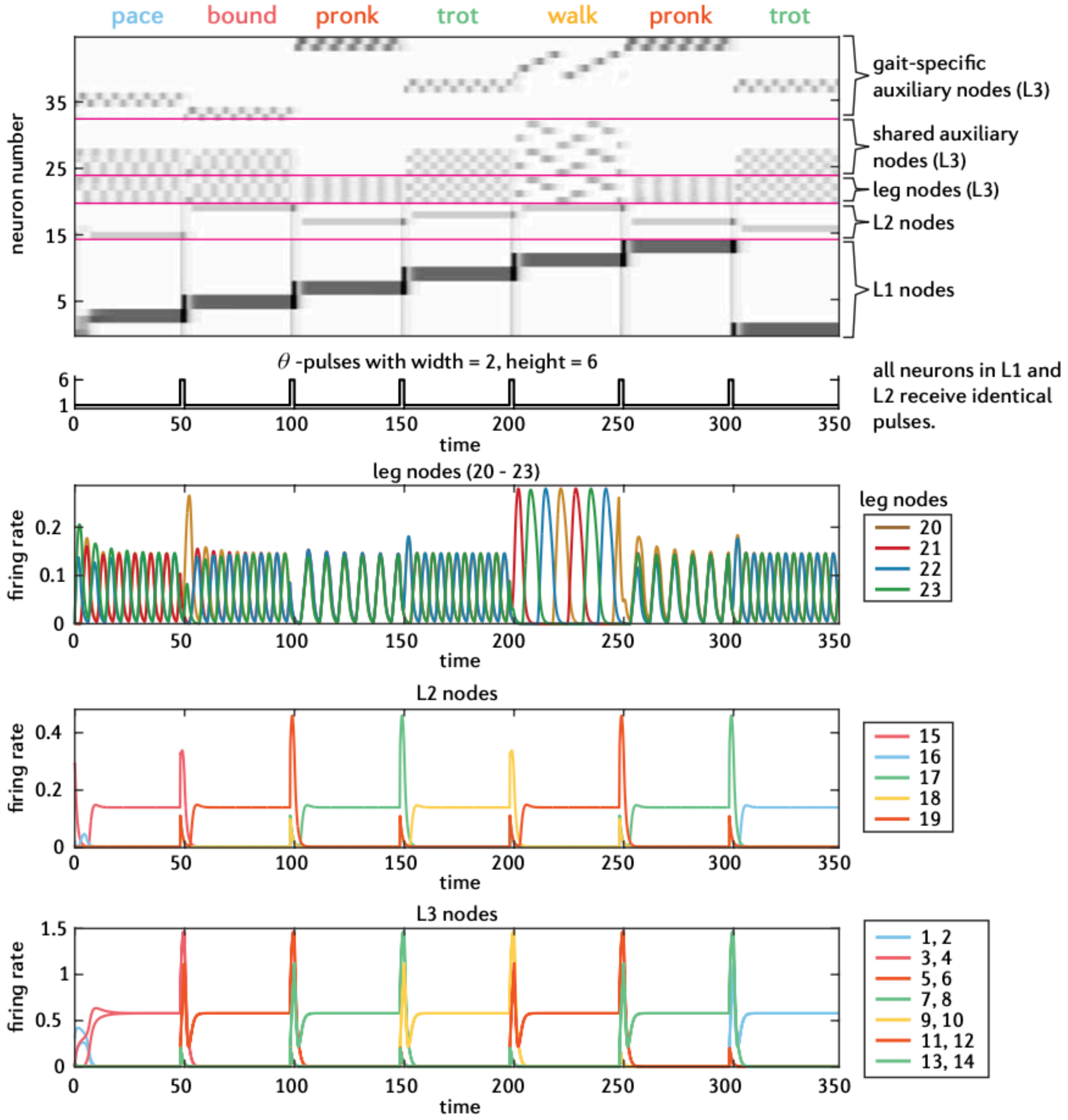
Juliana's quadruped gaits paper

Juliana Londoño
Alvarez

Idea: Using cyclic unions, build a single network that encodes 5 quadruped gaits, and couple it to a "counter" network allowing the network to step through a sequence of gaits via identical input pulses.







Attractor-based models for sequences and pattern generation in neural circuits

Juliana Londono Alvarez, Katie Morrison, Carina Curto

doi: https://doi.org/10.1101/2025.03.07.642121

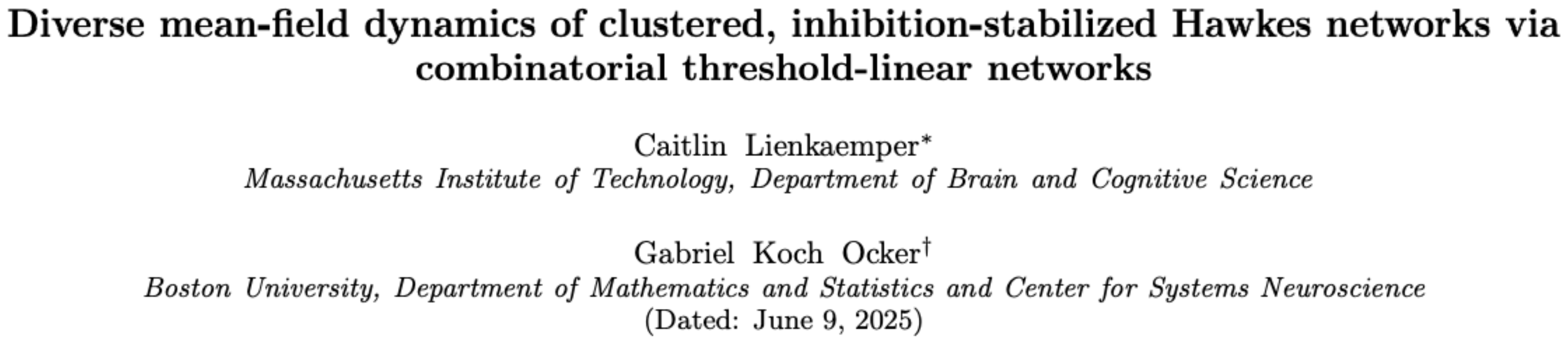
https://www.biorxiv.org/content/10.1101/2025.03.07.642121v2

Caitlin's mean field CTLNs paper

Caitlin Lienkaemper

Idea: CTLNs provide a mean field reduction of a spiking neural network model where each node in the CTLN represents a population in a large clustered network architecture.

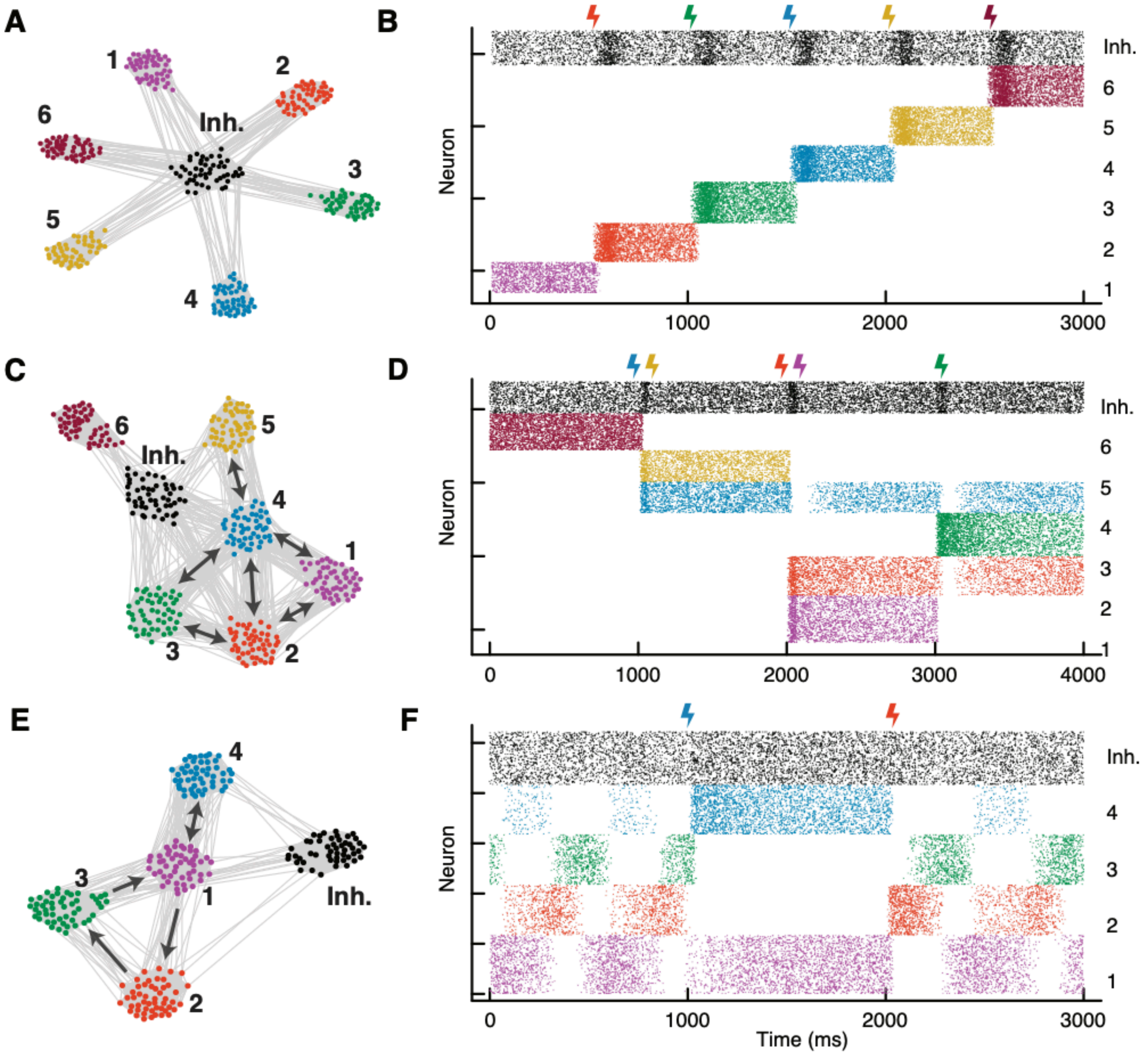
https://arxiv.org/abs/2506.06234



Diverse mean-field dynamics of clustered, inhibition-stabilized Hawkes networks via combinatorial threshold-linear networks

Caitlin Lienkaemper*
Massachusetts Institute of Technology, Department of Brain and Cognitive Science

Gabriel Koch Ocker†
Boston University, Department of Mathematics and Statistics and Center for Systems Neuroscience
(Dated: June 9, 2025)

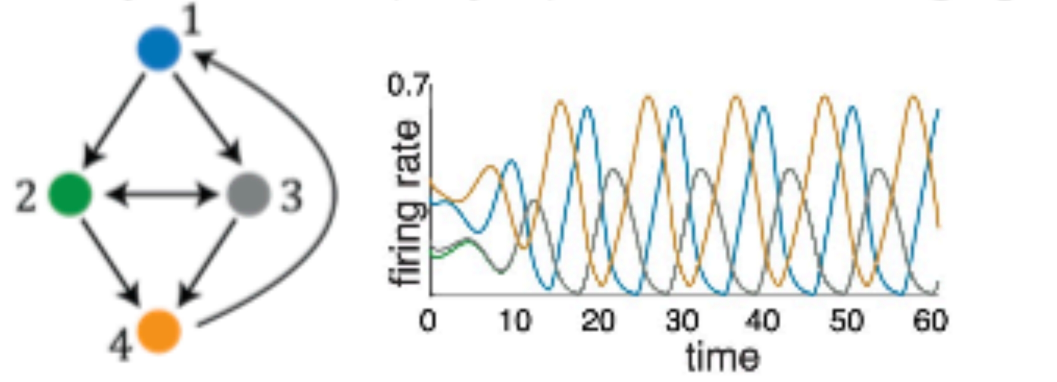


Zelong Li

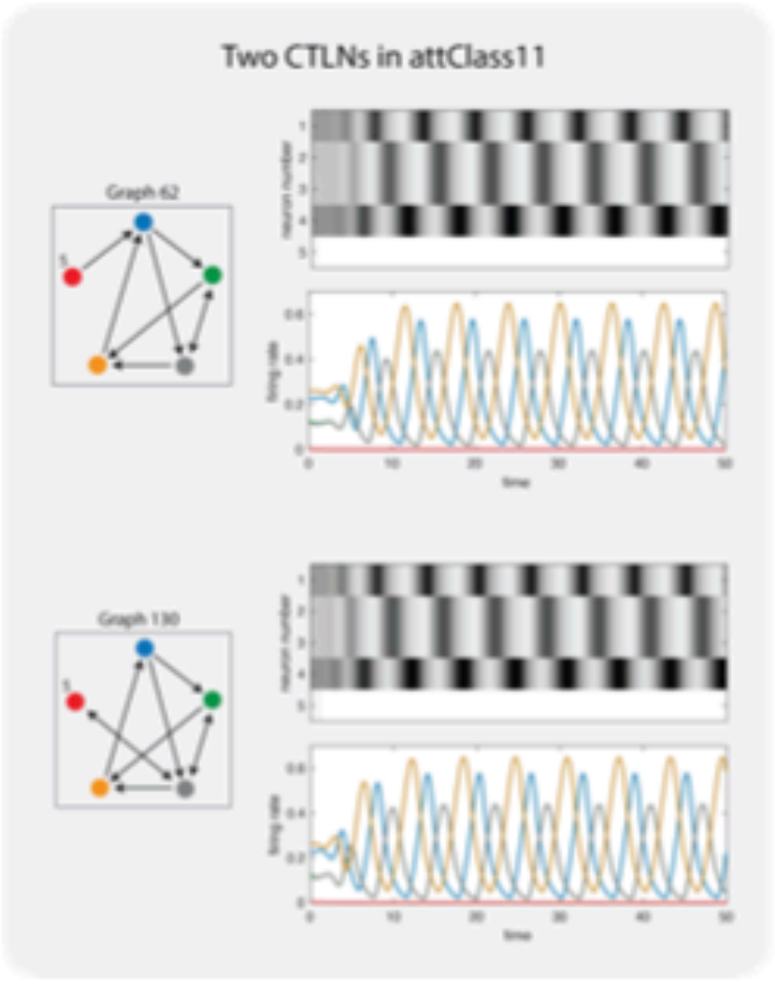
Check out Zelong's poster this evening !

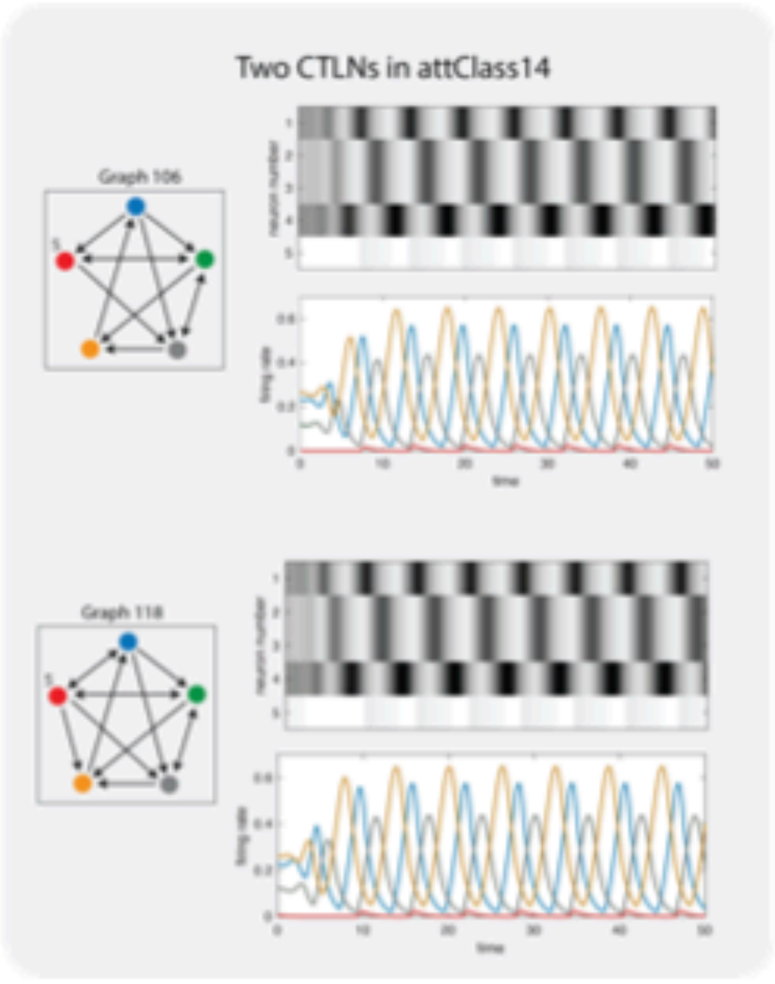
Observation of different CTLNs can be same attractor

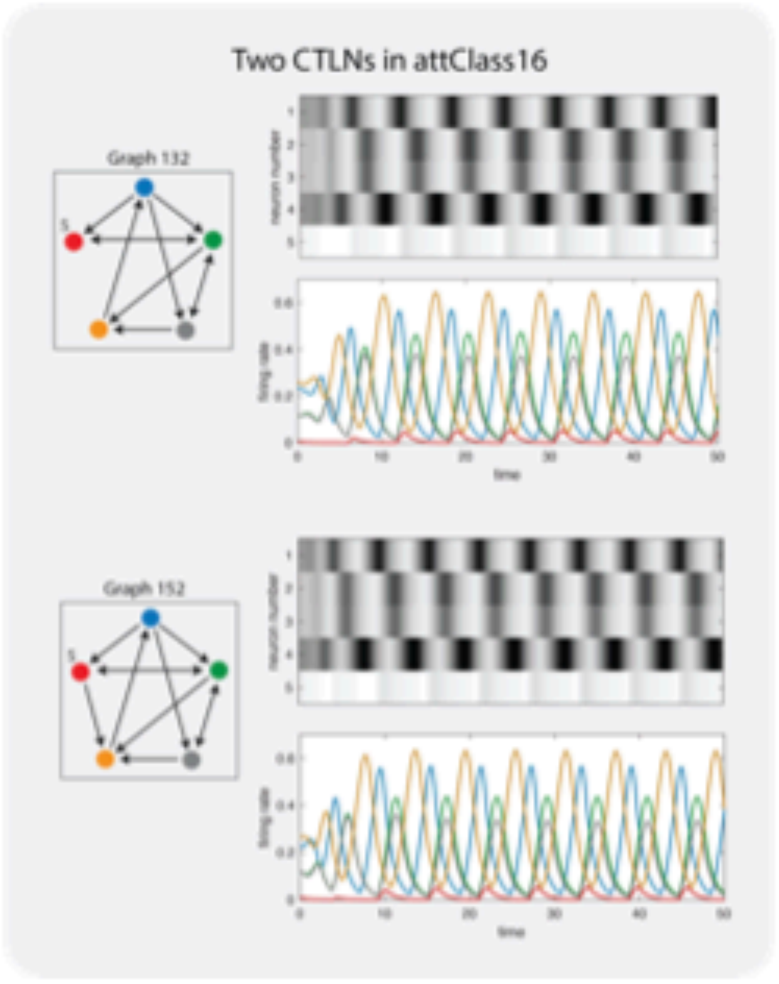
A 4-cyclic union (4-cycu) used as a based graph



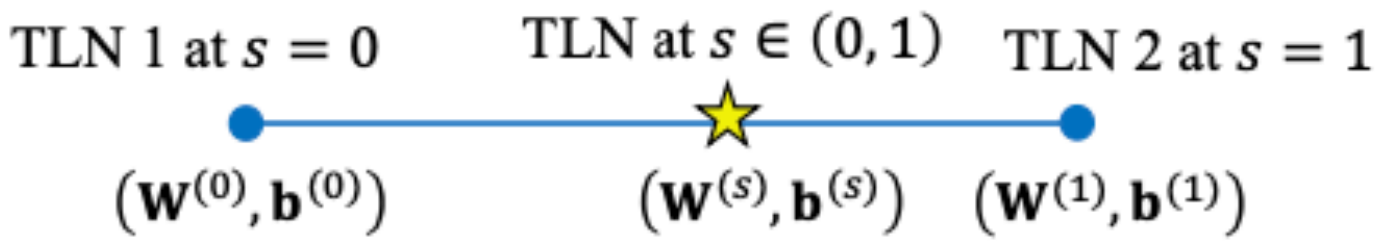
Out of 9608 non-isomorphic directed graphs on 5 neurons there are 1053 graphs containing dynamical attractors ([11]), which can be further organized into different structural attractor families.







All convex combinations of TLNs with the same attractor also have that attractor



TLN 1 at $s = 0$ TLN at $s \in (0,1)$ TLN 2 at $s = 1$

$(\mathbf{W}^{(0)}, \mathbf{b}^{(0)})$ $(\mathbf{W}^{(s)}, \mathbf{b}^{(s)})$ $(\mathbf{W}^{(1)}, \mathbf{b}^{(1)})$

TLN vector field:

$$\frac{dx}{dt} = v^{(s)}(\boldsymbol{x}) \coloneqq -\boldsymbol{x} + \left[\mathbf{W}^{(s)}x + \mathbf{b}^{(s)}\right]_+$$

$$\mathbf{W}^{(s)} = (1-s)\mathbf{W}^{(0)} + s\mathbf{W}^{(1)}$$
$$\mathbf{b}^{(s)} = (1-s)\mathbf{b}^{(0)} + s\mathbf{b}^{(1)}$$

Lemma 1 Suppose there exists a certain point $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that the vector fields match:

$$v^{(1)}(\boldsymbol{x}_0) = v^{(0)}(\boldsymbol{x}_0).$$

Then for all $s \in [0,1]$,

$$v^{(s)}(\boldsymbol{x}_0) = v^{(0)}(\boldsymbol{x}_0).$$

Corollary 2 If a fixed point of both $(\mathbf{W}^{(0)}, \mathbf{b}^{(0)})$ and $(\mathbf{W}^{(1)}, \mathbf{b}^{(1)})$, then it is a fixed point for all convex combinations $(\mathbf{W}^{(s)}, \mathbf{b}^{(s)})$.

Corollary 3 If $\boldsymbol{x}(t)$, for $t \in (t_0, t_1)$ is a trajectory of both $(\mathbf{W}^{(0)}, \mathbf{b}^{(0)})$ and $(\mathbf{W}^{(1)}, \mathbf{b}^{(1)})$, then it is also a trajectory for all convex combinations $(\mathbf{W}^{(s)}, \mathbf{b}^{(s)})$.

Safaan's poster was yesterday — but his PhD thesis is out on the arXiv:

Safaan Sadiq

arXiv > q-bio > arXiv:2508.07471

Quantitative Biology > Neurons and Cognition

[Submitted on 10 Aug 2025]

Modeling bias in decision-making attractor networks

Safaan Sadiq

https://arxiv.org/abs/2508.07471

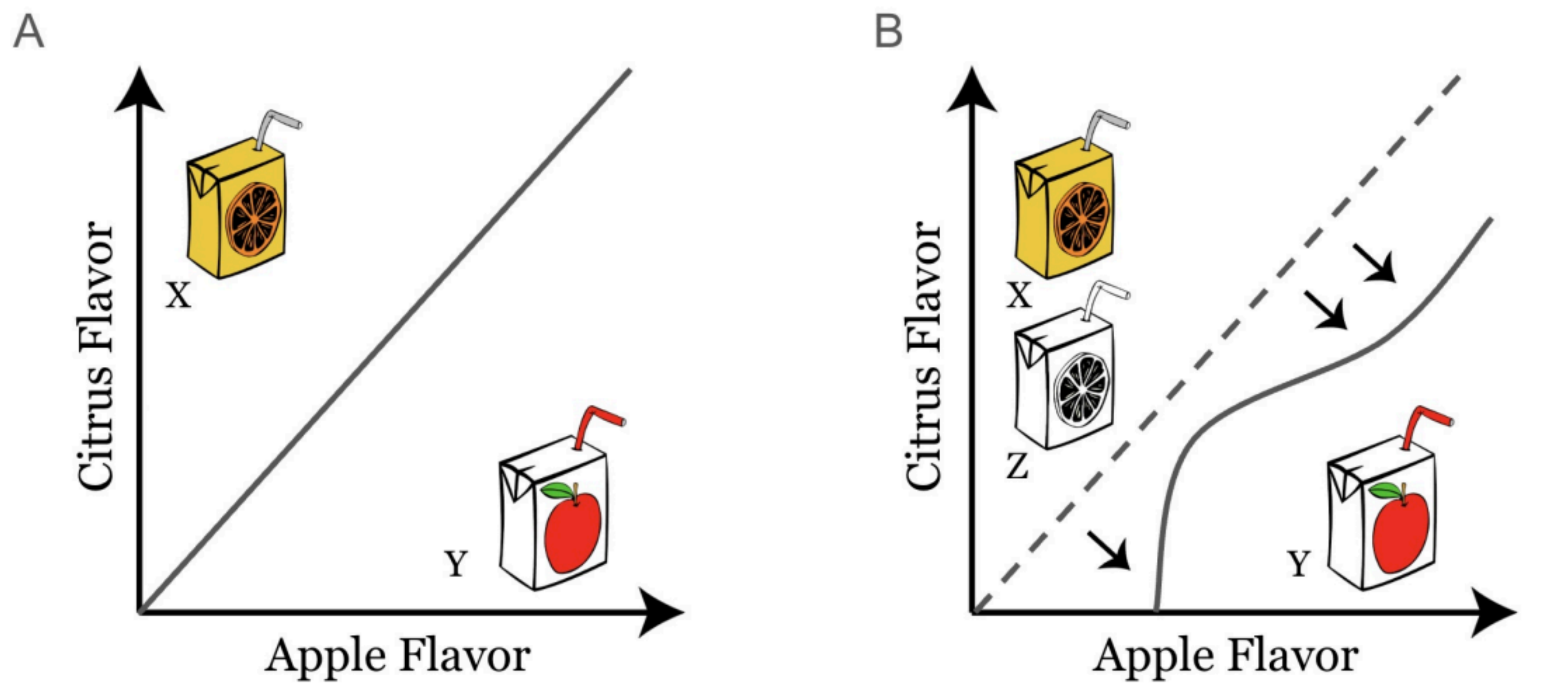
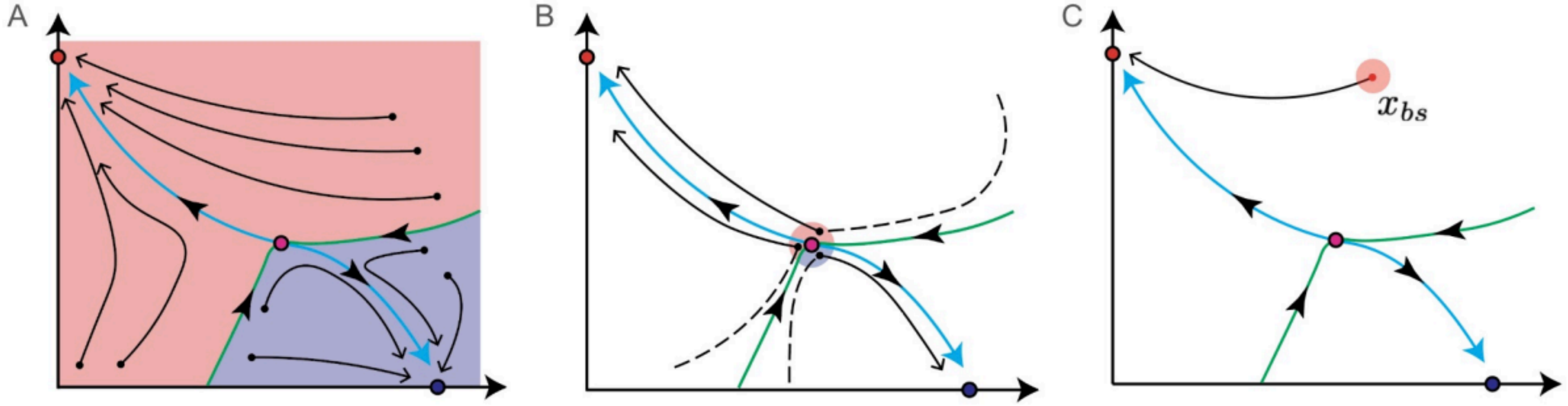


Figure 1.1. The Decoy Effect. (A) Orange juice "X" and apple juice "Y" have distinct flavors and depending upon preferences toward one or the other, either may be chosen roughly equivalently. The diagonal decision boundary reflects the 50% – 50% split. (B) The presence of a poorer quality orange juice "Z" does not add a true choice, but it increases the number of situations where "X" is the preferred choice, shifting the decision boundary.



Thank you!


Katie Morrison

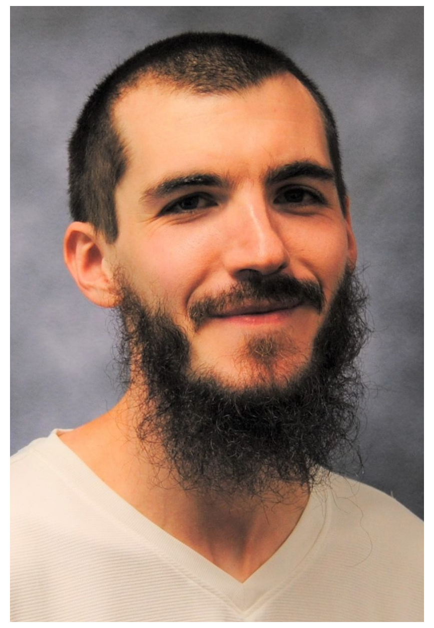

Caitlyn Parmelee


Chris Langdon


Nicole Sanderson


Zelong Li


Jency (Yuchen) Jiang


Jesse Geneson


Caitlin Lienkaemper


Juliana Londoño Alvarez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Other Collaborators:


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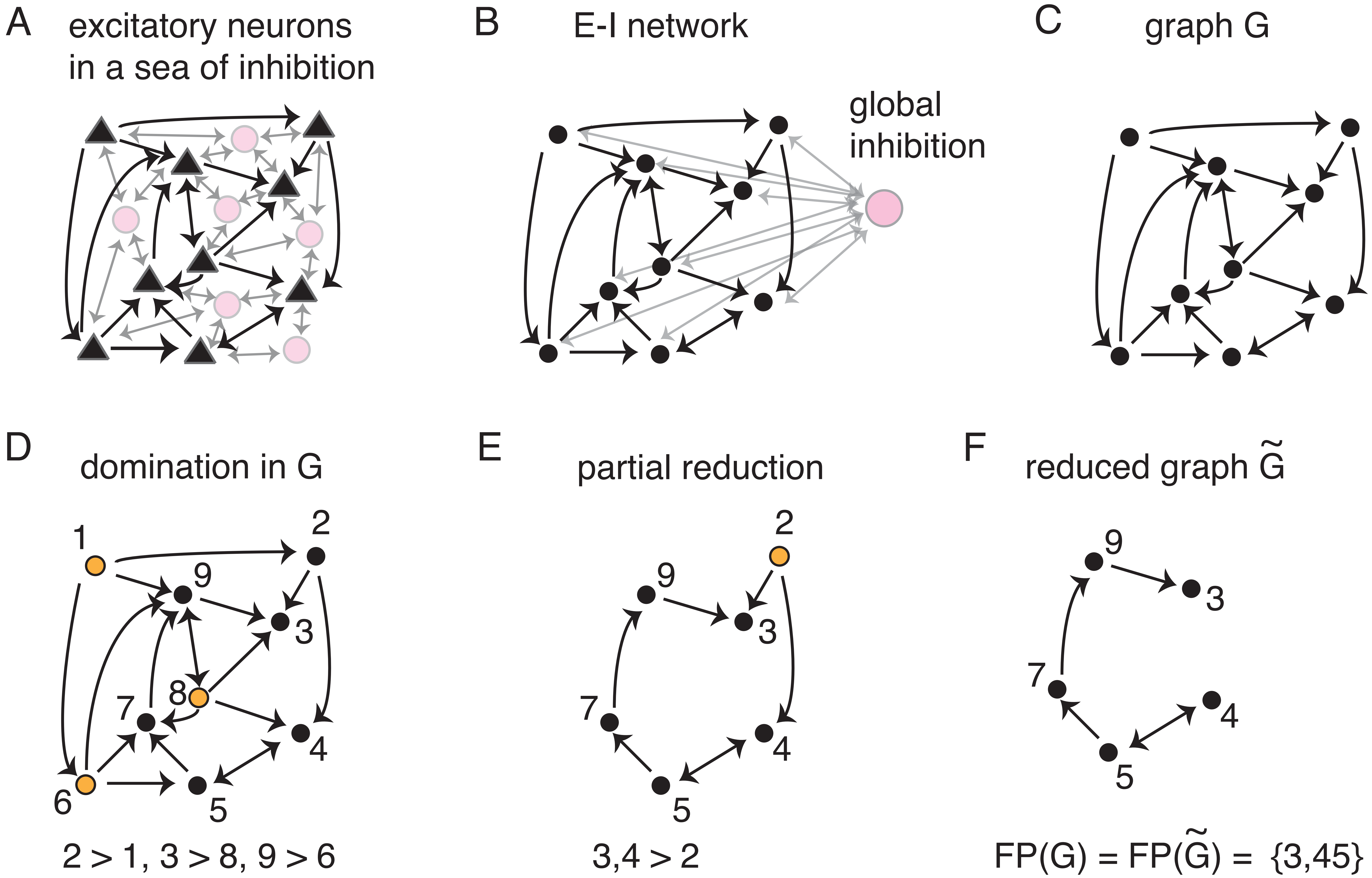
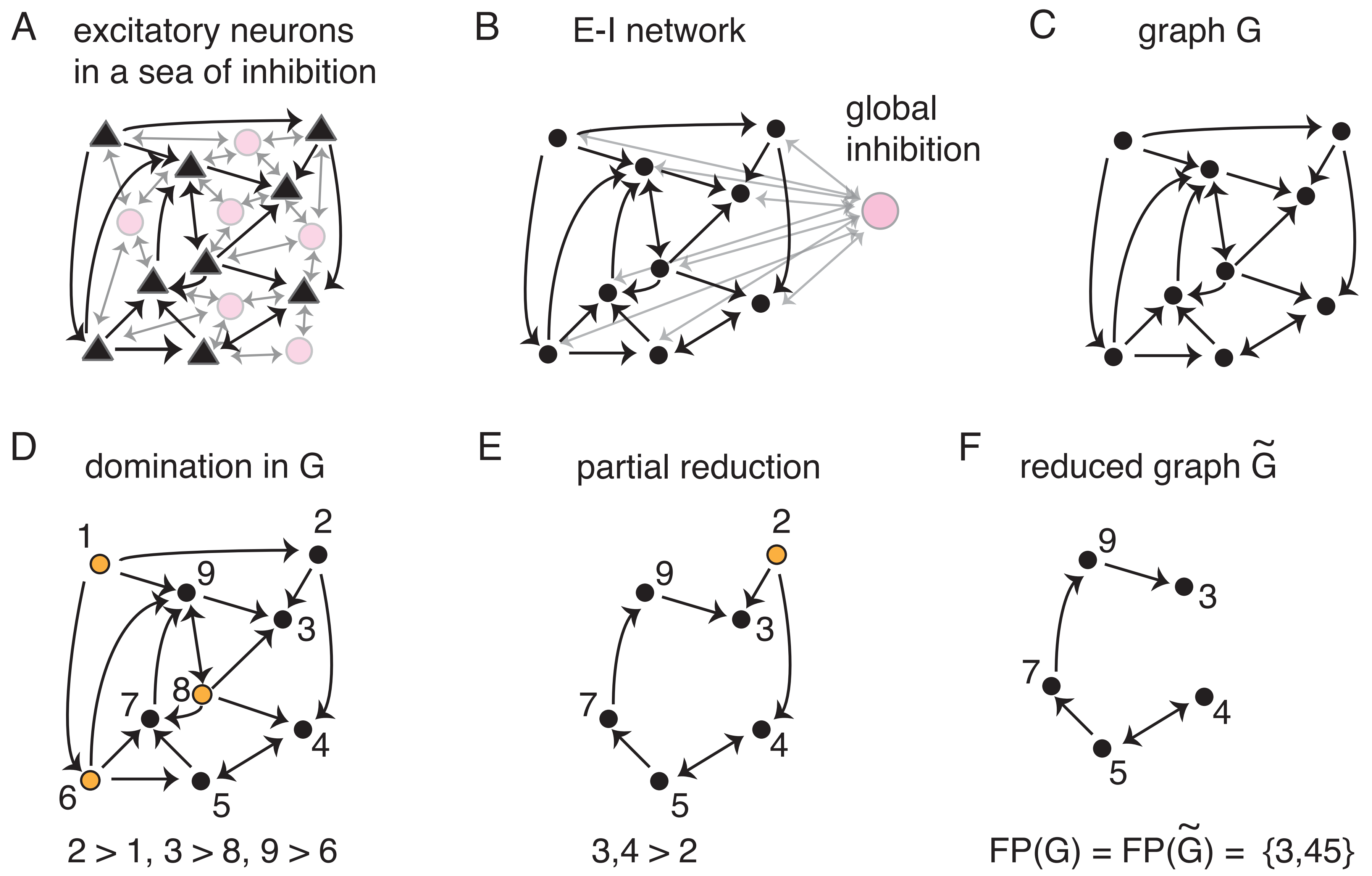
Johns Hopkins University Applied Physics Laboratory,
Research & Exploratory Development Department

Thank you!


A
excitatory neurons in a sea of inhibition
B
E-I network
global inhibition
C
graph G
D
domination in G
1
2
9
3
8
7
4
6
5
2 > 1, 3 > 8, 9 > 6
E
partial reduction
2
9
3
7
4
5
3,4 > 2
F
reduced graph G̃
9
3
7
4
5
FP(G) = FP(G̃) = {3,45}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7
4
5
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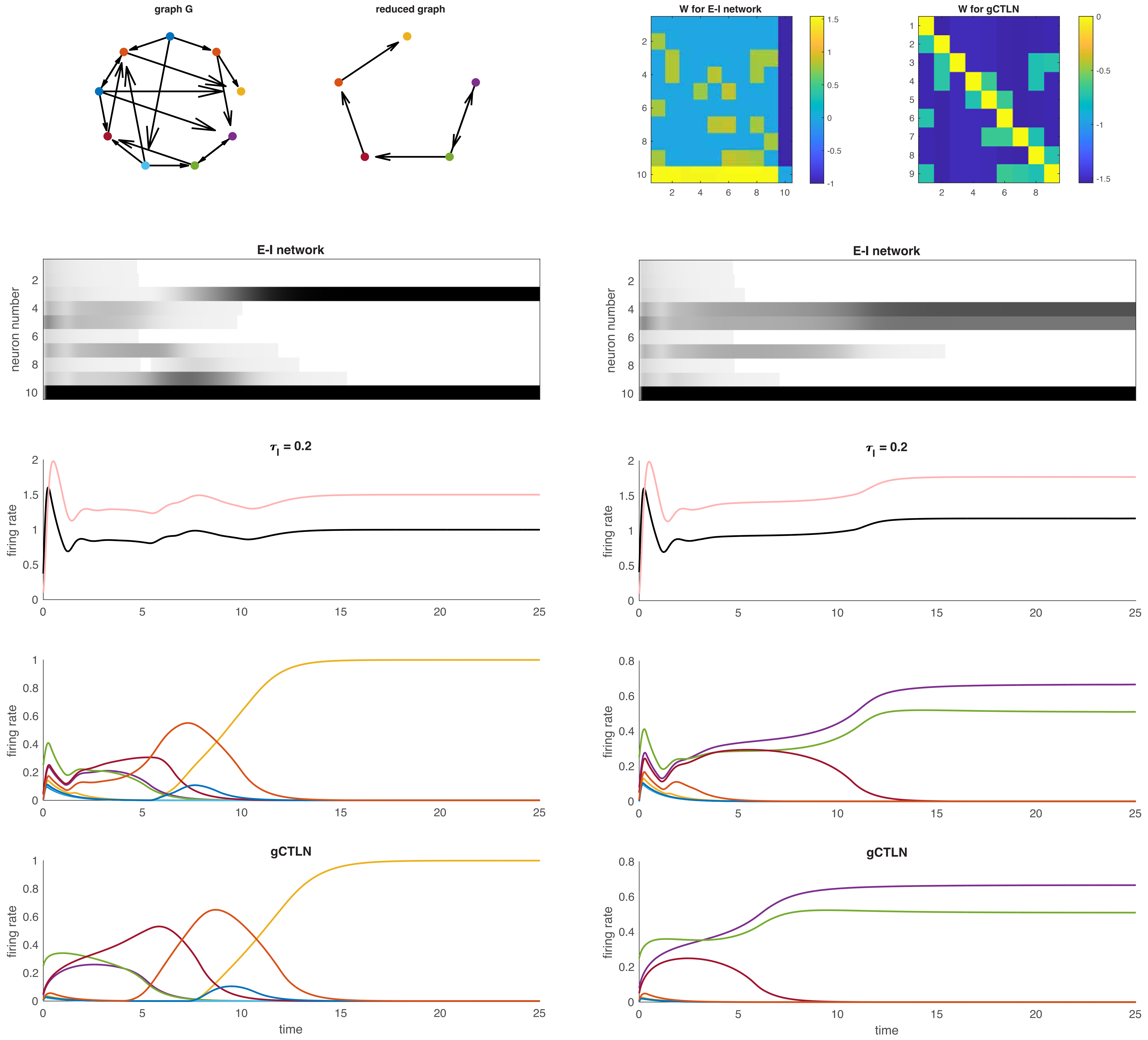
graph G
reduced graph
W for E-I network
W for gCTLN
E-I network
neuron number
τ_I = 0.2
firing rate
gCTLN
time
0
5
10
15
20
25