Multi Network RNA Velocity

Boya Hou

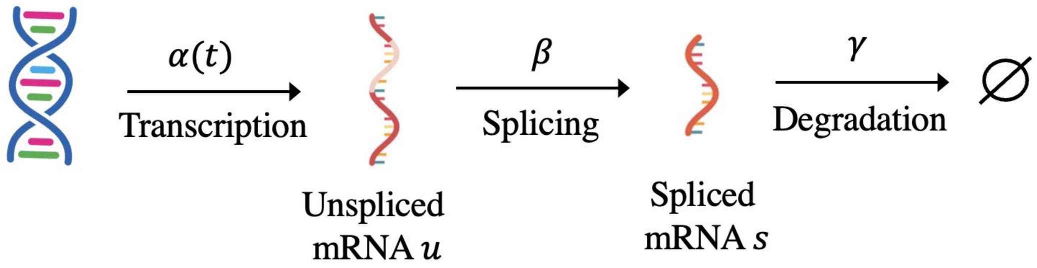
University of Illinois Urbana-Champaign

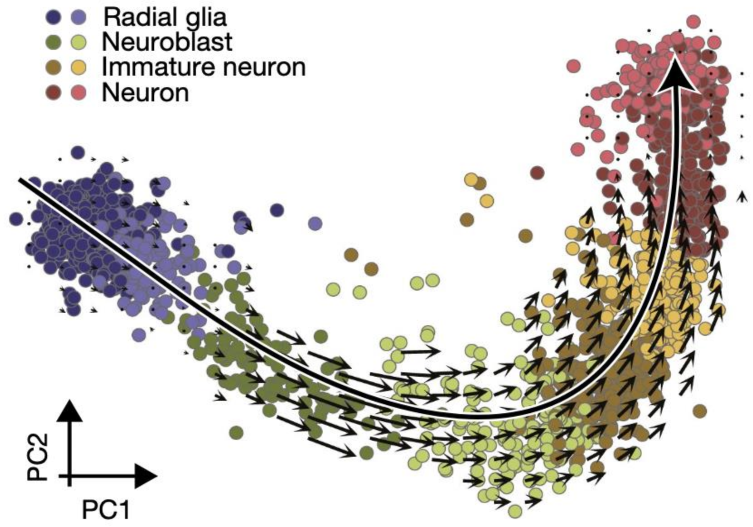
Joint work with Prof. Maxim Raginsky, Dr. Abhishek Pandey, Prof. Olgica Milenkovic

NITMB MathBio Convergence Conference

August 11th, 2025

DNA Transcription



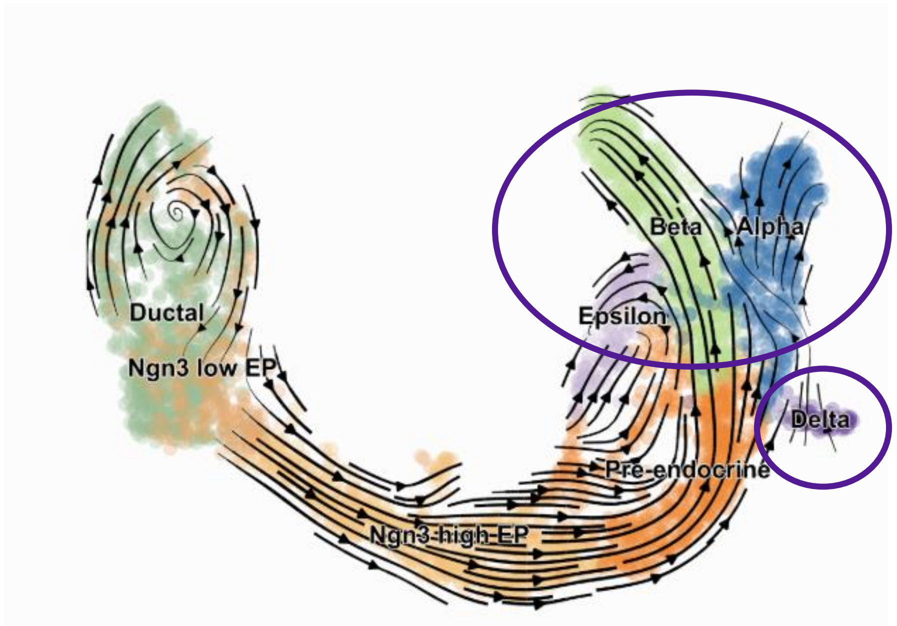


La Manno et al. 2018

Unspliced $$\frac{du}{dt} = \alpha(t) - \beta u(t)$$

Spliced $$\frac{ds}{dt} = \beta u(t) - \gamma s(t)$$

RNA velocity (La Manno et al., 2018) $$v(t) = \frac{ds}{dt} = \beta u(t) - \gamma s(t)$$



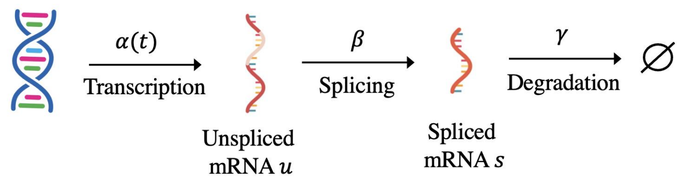
Bergen et al. 2020, pancreas .

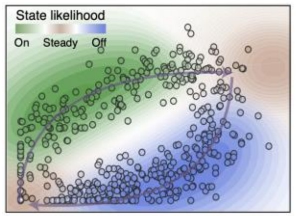
La Manno et al. "RNA velocity of single cells." *Nature* 560.7719 (2018): 494-498.

Bergen, Volker, et al. "Generalizing RNA velocity to transient cell states through dynamical modeling." Nature biotechnology 38.12 (2020): 1408-1414.

Gorin, G., Fang, M., Chari, T., & Pachter, L. (2022). RNA velocity unraveled. *PLOS Computational Biology*, *18*(9), e1010492.

Models of RNA Velocity



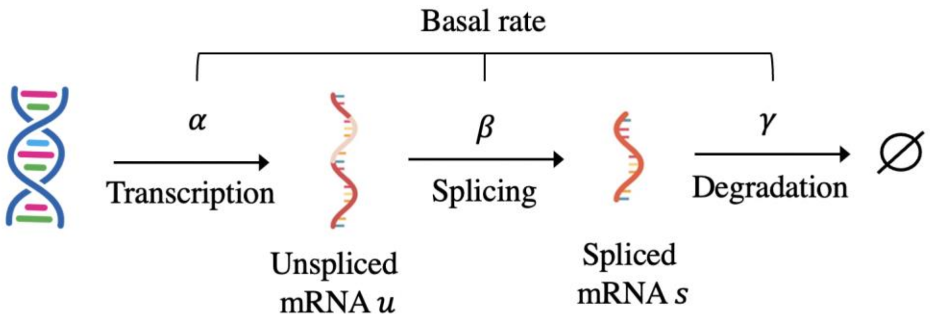
Model	Model Description	Limitation
Velocyto [Manno et al. 2018]	Steady-state ratio of unspliced to spliced RNA: $\frac{\gamma}{\beta}$. Velocity: $v = u - \frac{\gamma}{\beta} s$.	Assumes a steady-state model.
scVelo [Bergen et al. 2020] 	Full dynamical model: $u(t) = u_0 e^{-\beta\tau} + \frac{\alpha^{(k)}}{\beta}\left(1 - e^{-\beta\tau}\right), \quad \tau = t - t_0^{(k)}$ $s(t) = s_0 e^{-\gamma\tau} + \frac{\alpha^{(k)}}{\gamma}\left(1 - e^{-\gamma\tau}\right) + \frac{\alpha^{(k)} - \beta u_0}{\gamma - \beta}\left(e^{-\gamma\tau} - e^{-\beta\tau}\right).$	Treats each gene independently and regulatory relationships are ignored.
TFVelo [Li et al. 2024]	Transcription factor-aware: $y_g(t) = \alpha_g \sin(\omega_g t + \theta_g) + \beta_g,$ $\frac{dy_g(t)}{dt} = W_g X_g(t) - \gamma_g y_g(t).$	• Assume a specific behavioral form (sine function). • Does not explicitly integrate GRNs as transcription rate controllers.

La Manno et al. "RNA velocity of single cells." *Nature* 560.7719 (2018): 494-498.

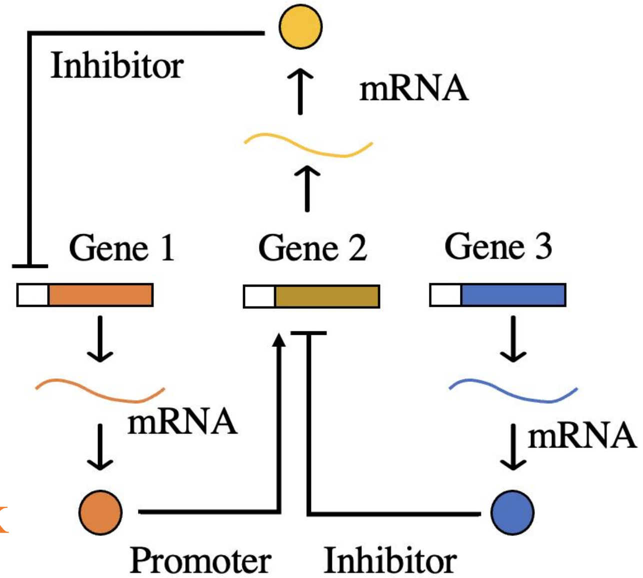
Bergen, Volker, et al. "Generalizing RNA velocity to transient cell states through dynamical modeling." Nature biotechnology 38.12 (2020): 1408-1414.

Li, Jiachen, et al. "TFvelo: gene regulation inspired RNA velocity estimation." Nature Communications 15.1 (2024): 1387.

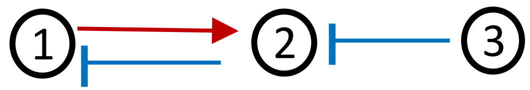
Our Model for Network RNA Velocity



Gene Regulatory Network



Unspliced $$\frac{du_i^g}{dt} = \alpha_i^g \frac{\kappa + \sum_{q=1}^{n_g} W_{gq}^{+} s_i^q(t)}{\kappa + \sum_{q=1}^{n_g} W_{gq}^{-} s_i^q(t)} - \beta_i^g u_i^g(t),$$



Spliced $$\frac{ds_i^g}{dt} = \beta_i^g u_i^g(t) - \gamma_i^g s_i^g(t) + \frac{c}{n_c} \sum_{j=1}^{n_c} a_{ij} \left(s_j^g(t) - s_i^g(t) \right)$$

A graph model of GRN

Intercellular Network (not in this talk)

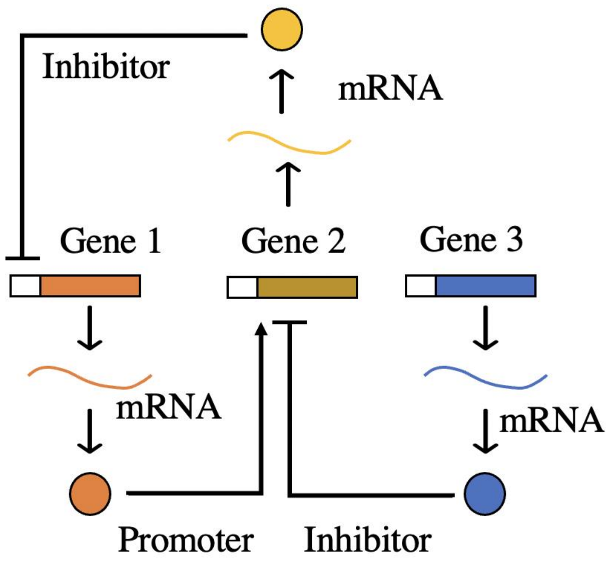
Goal: Study network RNA velocity and targeted drug interventions (in collaboration with AbbVie).

Incremental Gain and Regulations in GRNs

Unspliced $$\frac{du^g}{dt} = \alpha^g \frac{\kappa + \sum_{q=1}^{n_g} W_{gq}^{+} s^q(t)}{\kappa + \sum_{q=1}^{n_g} W_{gq}^{-} s^q(t)} - \beta^g u^g(t),$$

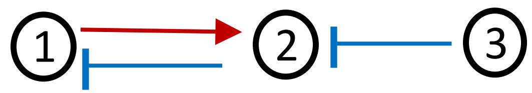
Spliced $$\frac{ds^g}{dt} = \beta^g u^g(t) - \gamma^g s^g(t).$$

$$R_g(s) := \frac{\kappa + \sum_{q=1}^{n_g} W_{gq}^{+} s^q(t)}{\kappa + \sum_{q=1}^{n_g} W_{gq}^{-} s^q(t)} = \frac{N_g(s)}{D_g(s)}$$



The incremental gain of R_g due to a change from s^q to $\hat{s}^q$ is

$$\frac{R_g(s) - R_g(\hat{s})}{s^q - \hat{s}^q} = \frac{DW_{gq}^{+} - NW_{gq}^{-}}{DD'}$$

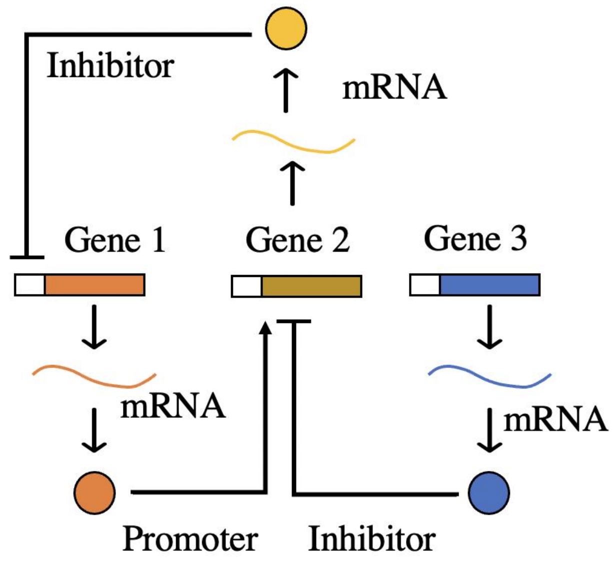


A graph model of GRN

The incremental gain will be:

positive if $W_{gq}^{+} > 0$ (and thus $W_{gq}^{-} = 0$), and negative if $W_{gq}^{-} > 0$ (and thus $W_{gq}^{+} = 0$) .

Existence of Steady States



Unspliced $$\frac{du^g}{dt} = \alpha^g \frac{\kappa + \sum_{q=1}^{n_g} W_{gq}^{+} s^q(t)}{\kappa + \sum_{q=1}^{n_g} W_{gq}^{-} s^q(t)} - \beta^g u^g(t),$$

Spliced $$\frac{ds^g}{dt} = \beta^g u^g(t) - \gamma^g s^g(t).$$

For a nonlinear system $\frac{dx}{dt} = f(x)$, a point x_e is a steady state if $f(x_e) = 0$.

Theorem 1:

Suppose that $\beta_g > 0$ and $\gamma_g > 0$ for all genes g. Let $C = \max_g \sum_{q=1}^{n_g} W_{gq}^{+}$ and $\xi = \max_g \frac{\alpha^g}{\gamma^g}$.

The networked dynamics admits a steady state (u^*, s^*) if $\kappa \geq C\xi$.

Stability Analysis

- What does it mean for a system to be stable?

  Suppose a system has a steady state (u^*, s^*). If you slightly perturb the system, does it:

  Return to the steady state (Stable) ? Or drift away over time (Unstable)?

- System is globally asymptotically stable if for every trajectory $x(t)$, we have $x(t) \to x_e$ as $t \to \infty$.

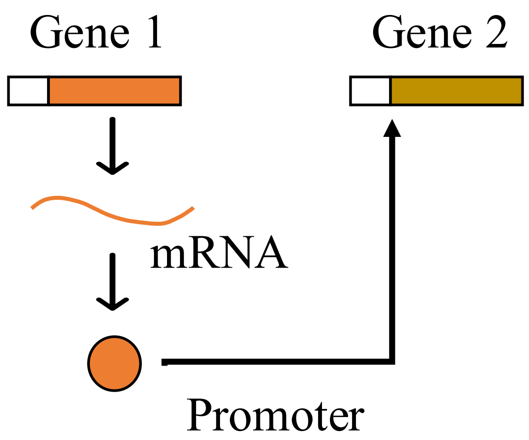
Single Gene Case:

$$\frac{d}{dt}\begin{bmatrix} u \\ s \end{bmatrix} = \underbrace{\begin{bmatrix} -\beta & 0 \\ \beta & -\gamma \end{bmatrix}}_{=A} \begin{bmatrix} u \\ s \end{bmatrix} + \begin{bmatrix} \alpha \\ 0 \end{bmatrix}.$$

Linear system of the form $\dot{x} = Ax$, globally asymptotic stable iff $\mathcal{R}e(\lambda(A)) < 0$.

Eigenvalues of A: $\lambda_1 = -\beta < 0$, $\lambda_2 = -\gamma < 0$. Thus, the system is always stable.

Stability of Promoter-Only GRNs



A pure positive regulation network

$$\frac{du^g}{dt} = \alpha^g \left(\kappa + \sum_{q=1}^{n_g} W_{gq}^{+} s^q(t) \right) - \beta^g u^g(t)$$
$$\frac{ds^g}{dt} = \beta^g u^g(t) - \gamma^g s^g(t).$$

Linear system, we can write it as $\dot{x} = Ax$, $x = \binom{u^g}{s^g}$.

The linear system $\dot{x} = Ax$ is globally asymptotic stable iff $\mathcal{R}e(\lambda(A)) < 0$.

Lemma 1: Suppose the condition of Theorem 1 holds s.t. a steady state exists.

When there is no inhibitor, i.e., $W_{gq}^{-} = 0$ for all genes, the networked dynamics is stable

if, $\gamma_g > \beta_g > \alpha_g \sum_{h=1}^{n_g} W_{gh}^{+}$, for all g.

General Case: Understanding Stability via Lyapunov Functions

$$\frac{du^g}{dt} = \alpha^g \frac{\kappa + \sum_{q=1}^{n_g} W_{gq}^{+} s^q(t)}{\kappa + \sum_{q=1}^{n_g} W_{gq}^{-} s^q(t)} - \beta^g u^g(t),$$

$$\frac{ds^g}{dt} = \beta^g u^g(t) - \gamma^g s^g(t).$$

Goal: make conclusions about trajectories of a system $\dot{x} = f(x)$ (e.g., globally asymptotically stable) without finding the trajectories (i.e., solving the differential equations).

A nonlinear system $\dot{x} = f(x)$.

A Lyapunov global asymptotic stability theorem:

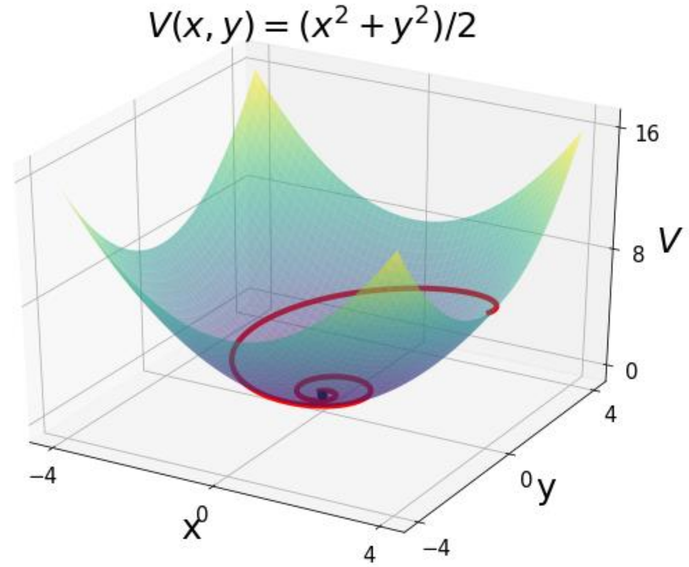
Suppose there exists a function $V: R^n \rightarrow R$ that is positive definite,

i.e., $V(x) \geq 0$ for all x, $V(x) = 0$ iff $x = x_e$,

and $V(x) \rightarrow \infty$ whenever $\|x\| \rightarrow \infty$.

In addition, $\dot{V}(x) < 0$ for all $x \neq x_e$, and $\dot{V}(x_e) = 0$.

Then, every trajectory of $\dot{x} = f(x)$ converges to x_e as $t \rightarrow \infty$.



We call V a Lyapunov function, which can be thought of as a generalized energy function.

Stability of Network RNA Velocity

$$\frac{du^g}{dt} = \alpha^g \frac{\kappa + \sum_{q=1}^{n_g} W_{gq}^{+} s^q(t)}{\kappa + \sum_{q=1}^{n_g} W_{gq}^{-} s^q(t)} - \beta^g u^g(t),$$
$$\frac{ds^g}{dt} = \beta^g u^g(t) - \gamma^g s^g(t).$$

Theorem 2: Suppose the condition of Theorem 1 holds s.t. a steady state exists.

Consider a positive semi-definite function as a candidate Lyapunov function,

$$V(u, s) := \frac{1}{2} \|u - u^*\|_2^2 + \frac{1}{2} \|s - s^*\|_2^2$$

- Suppose there is sufficient negative regulation in the network, i.e., there is $\delta > 0$ s.t.

$$\min_g \sum_{q=1}^{n_g} W_{gq}^{-} s^q \geq \delta \left(\sum_{q=1}^{n_g} s^q \right).$$

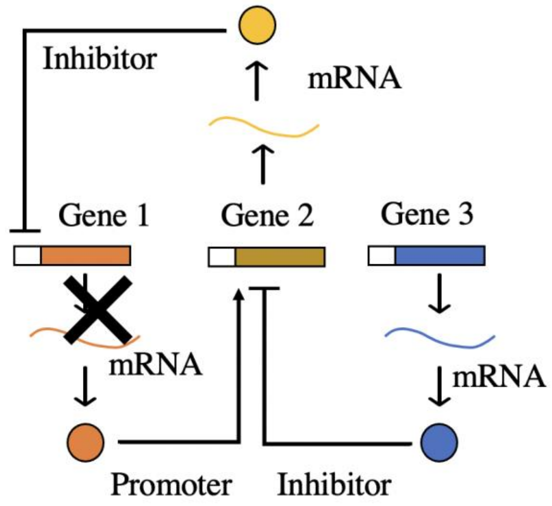
- If for some constant ω that depends on the GNR, the parameters satisfy

$$\beta^g > \frac{\omega \|\alpha\|}{2}, \quad \gamma^g > \frac{\omega \|\alpha\|}{2} + \frac{(\beta^g)^2}{4(\beta^g - \frac{\omega\|\alpha\|}{2})}.$$

Then, $V(u, s)$ is a Lyapunov function, and (u^*, s^*) is globally asymptotically stable.

Overview of Targeted Drug Intervention

- Let $z^q(t)$ be the control input (drug intervention) targeting gene q.



$$\frac{du^g}{dt} = \alpha^g \underbrace{\frac{\kappa + \sum_{p\neq q}^{n_g} W_{gp}^{+} s^p(t) + z^q(t) W_{gq}^{+} s^q(t)}{\kappa + \sum_{p=1}^{n_g} W_{gp}^{-} s^p(t)}}_{:=R_g^{\circ}(z^q,s)} - \beta^g u^g(t),$$

$$\frac{ds^g}{dt} = \beta^g u^g(t) - \gamma^g s^g(t).$$

- Find the fastest possible intervention strategy that steers the dynamic to targeted profile s_f:

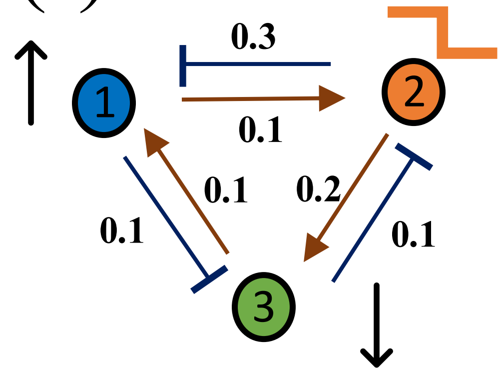
$$\begin{aligned} \min_{z^q} & \int_0^T 1 \ dt \\ \text{s.t. } & \dot{u} = \alpha R^{\circ}(z^q, s) - \beta u, \\ & \dot{s} = \beta u - \gamma s, \\ & u(0) = u_0, \quad s(0) = s_0, \quad s(T) = s_f, \\ & z^q(t) \in \mathbf{U}, \ \forall t \in [0, T]. \end{aligned}$$


Pontryagin’s Maximum Principle

Exemplary Simulation of Drug Intervention (I)

$$\frac{du^g}{dt} = \alpha^g \frac{\kappa + \sum_{q=1}^{n_g} W_{gq}^{+} s^q(t)}{\kappa + \sum_{q=1}^{n_g} W_{gq}^{-} s^q(t)} - \beta^g u^g(t),$$

$$\frac{ds^g}{dt} = \beta^g u^g(t) - \gamma^g s^g(t).$$

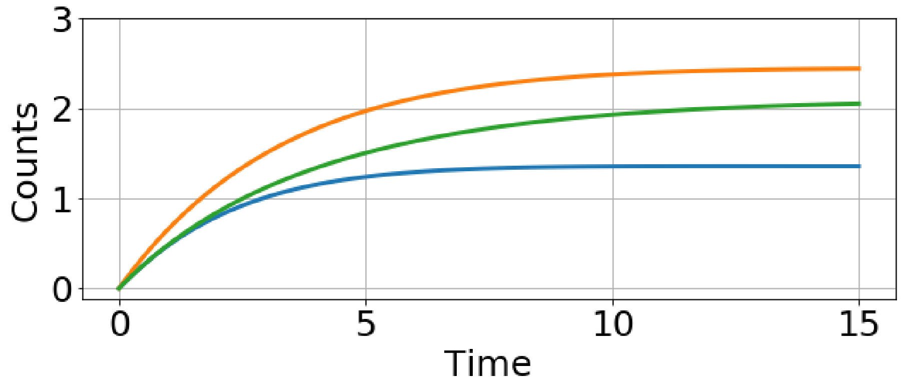


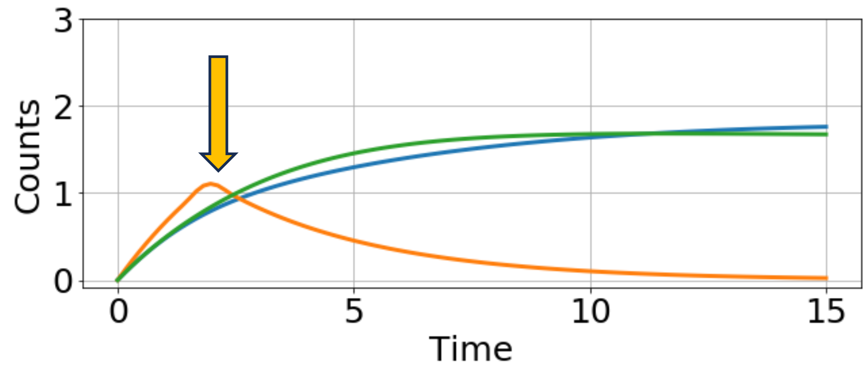
GRN

No intervention

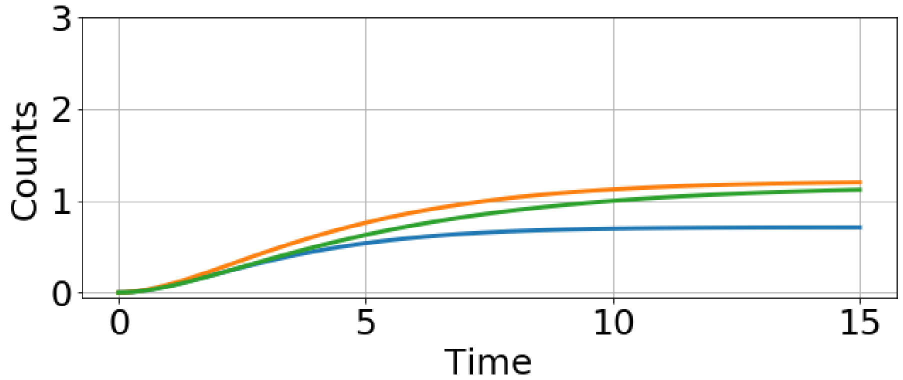
Knock out gene 2 at time t=2

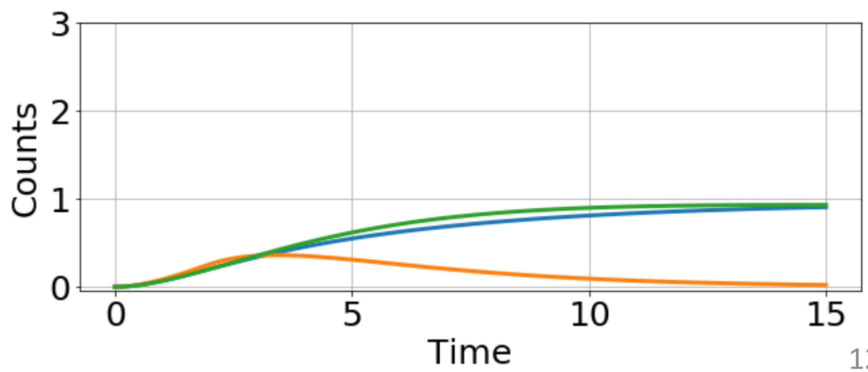
Unspliced $u(t)$





Spliced $s(t)$

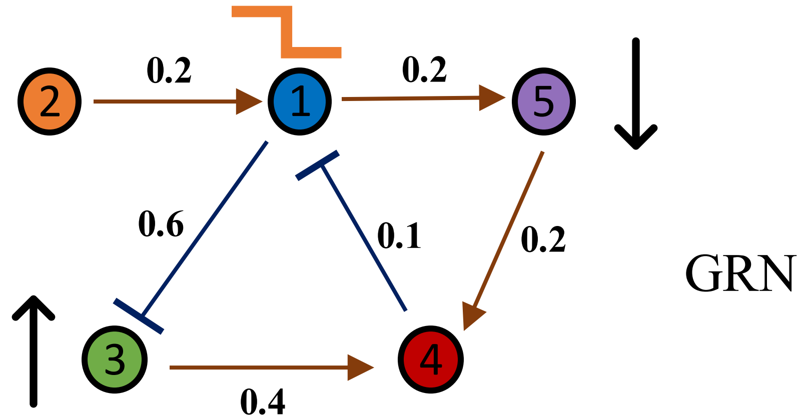




Exemplary Simulation (II)

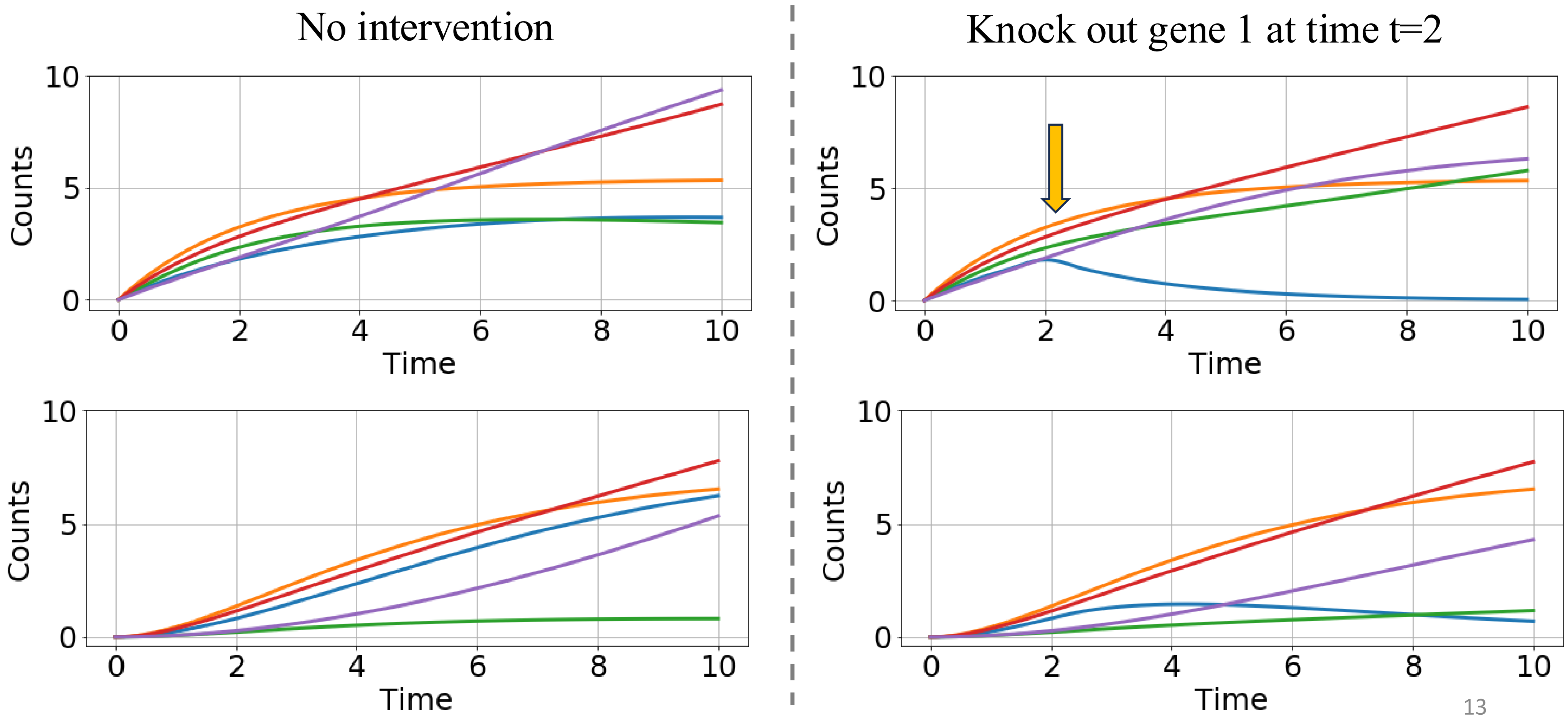
$$\frac{du^g}{dt} = \alpha^g \, \frac{\kappa + \sum_{q=1}^{n_g} W_{gq}^{+} s^q(t)}{\kappa + \sum_{q=1}^{n_g} W_{gq}^{-} s^q(t)} - \beta^g u^g(t),$$

$$\frac{ds^g}{dt} = \beta^g u^g(t) - \gamma^g s^g(t).$$



Unspliced $u(t)$

Spliced $s(t)$



More in the full manuscript …

- Incorporate cell-to-cell interaction via spatial transcriptomics

$$
\begin{aligned}
\frac{du_i^g}{dt} &= \alpha_i^g \, \frac{\kappa + \sum_{q=1}^{n_g} W_{gq}^{+} s_i^q(t)}{\kappa + \sum_{q=1}^{n_g} W_{gq}^{-} s_i^q(t)} - \beta_i^g u_i^g(t), \\
\frac{ds_i^g}{dt} &= \beta_i^g u_i^g(t) - \gamma_i^g s_i^g(t) + \frac{c}{n_c} \sum_{j=1}^{n_c} a_{ij} \left(s_j^g(t) - s_i^g(t) \right)
\end{aligned}
$$

- Examine how drug interventions may affect safety liability genes and design targeted drug intervention as controlled system.

Thank you !

boyahou2@illinois.edu

Scan me